Notes on Geometric Constraint Systems

F. C. Langbein

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Department of Computer Science
Cardiff University

Geometric Constraint Systems

Geometric Constraint System (G, O, t, C):

G	Finite set of sets representing types of geometric objects (set of all planes, set of all vertices, . . .)
O	Finite set of geometric objects (variables in the constraint system)
$t : O \to G$	Function assigning a type to each element of O (type constraints)
C	Finite set of geometric constraints; $c \in C$ is a subset of the product space $\Pi_{o \in O} t(o)$

Geometric Constraint Problems

- Function assigning a value to each object in O:

$$f : O \to \bigcup_{o \in O} t(o) \text{ s.t. } f(o) \in t(o)$$

- f is a solution iff

$$(f(o_1), \ldots, f(o_n)) \in S = \bigcap_{c \in C} c$$

- Geometric Constraint Problems:
  - ⋆ Solvability: at least one solution, cardinality of S
  - ⋆ Solutions: find solutions symbolically or numerically

Different Interpretations

- Different ways to interpret geometric constraints:

  - ⋆ Algebraic interpretation:
    All sets ($c \in C$, etc.) are described by equations of some type

  - ⋆ Geometric rule-based interpretation
    Constraints are represented as a set of rules and predicates

  - ⋆ Topological interpretation
    Consider the topological structure of the involved sets

Algebraic Interpretation

- Geometric constraint system as equation system
$$H(x) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_n(x) \end{bmatrix} = 0, \text{ where } H : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$
- Numerical solver:
  - ⋆ Polynomial equation system solver (e.g. Newton-Raphson, Homotopy)
  - ⋆ Optimisation methods (e.g. Quasi-Newton methods, evolutionary methods)

$\longrightarrow$ Yields numerical solution without exploiting geometric nature of the constraints, also no further information about constraint system structure is obtained

Symbolic Solver

- Symbolic Solver: Gröbner Bases / Polynomial Ideals
- Ideal of F where $K[x_1, \ldots, x_n]$ is ring of n-variable polynomials over coefficient field K:

$$I\langle F\rangle = \{h_1 f_1 + \cdots + h_n f_n : h_l \in K[x_1, \ldots, x_n]\}$$

- F is a basis for its ideal $I\langle F\rangle$, each basis has the same roots
- Transform original system into special Gröbner basis in its ideal to find roots, etc.

Geometric Rule-Based Interpretation

- Constraints are represented as a set of rules and predicates
- Employ rewrite rules to transform original representation into a construction sequence to find solution
- Rewrite rules represent the geometric knowledge of the solver
- Predicates describing constraints are transformed into predicates describing position, etc. of geometric objects

Topological Interpretation

- Consider the topology of the involved sets
- Assumptions:
  - ⋆ All geometry type spaces $g \in G$ are smooth, path-wise connected, compact manifolds
  - ⋆ The geometric constraints $c \in C$ are smooth, path-wise connected, compact sub-manifolds of the geometry product space $\Pi_{o \in O} t(o)$
    (only involved o have to be considered in product)
- S is created by intersecting these sub-manifolds
- Exploit local (dimension) and global structure of the manifolds to gather information about S

Geometric Types

- G is a set of manifolds whose elements are geometric objects of a single type:
  - ⋆ Set of all vertices: $\mathbb{R}^3$ (position)
  - ⋆ Set of all planes: $\mathbb{R}^1 \times \mathbb{P}^2$ (distance & direction)
  - ⋆ Set of all lines: 4-dim. manifold (see later)
  - $\ldots$
- t assigns a type to each object $o \in O$:
  - ⋆ Type constraint requiring that $f(o) \in t(o)$

Vertex Distance Constraint

- Geometric objects: $o_1, o_2 \in O$
- Geometric types: $t(o_1) = t(o_2) = \mathbb{R}^3$

$$(f(o_1), f(o_2)) \in \mathbb{R}^3 \times \mathbb{R}^3$$

- Constraint: constant distance λ between vertices o_1, o_2

$$(f(o_1), f(o_2)) \in \{(x_1, x_2) : \|x_1 - x_2\| = \lambda\} =: c$$

⋆ c is a sub-manifold of $\mathbb{R}^3 \times \mathbb{R}^3$

⋆ c is homeomorphic to

(1) $\mathbb{R}^3 \times \mathbb{S}^2$: Choose first vertex freely, then the 2nd vertex is determined by a direction

(2) $\mathbb{S}^2 \times \mathbb{R}^3$: Analogously, $o_1 \leftrightarrow o_2$

$\longrightarrow$ Two options to interpret (distribute) c as sub-manifold of $\mathbb{R}^3 \times \mathbb{R}^3$

Vertex on Plane Constraint

- Geometric objects: vertex $v \in O$, plane $p \in O$

$$(f(v), f(p)) \in \mathbb{R}^3 \times (\mathbb{R}^1 \times \mathbb{S}^2)$$

- Constraint: vertex v in plane p

$$(f(v), f(p)) \in \{(x_1, (x_2, x_3)) : x_3^t x_1 = x_2\} =: c$$

⋆ c as sub-manifold of $\mathbb{R}^3 \times (\mathbb{R}^1 \times \mathbb{S}^2)$ is homeomorphic to

(1) $\mathbb{R}^2 \times (\mathbb{R}^1 \times \mathbb{S}^2)$

- Choose an arbitrary plane
- Choose a point on the plane

(2) $\mathbb{R}^3 \times (\mathbb{R}^0 \times \mathbb{S}^2)$

- Choose an arbitrary point
- Choose a normal for the plane through the point

Multiple Constraints

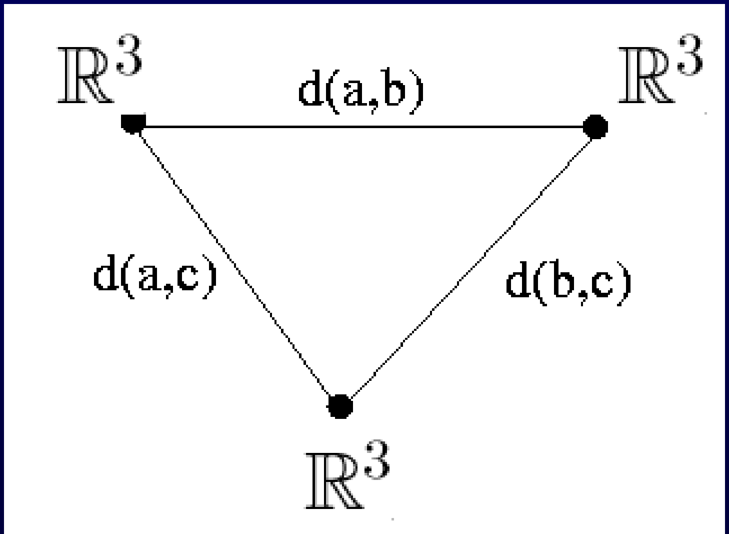
- In order to enforce multiple constraint the sub-manifolds have to be intersected
- Intersection of two "point on sphere":
  empty set, point, **circle**, sphere
- Intersection of two "planes through point":
  planes through line, planes through point
- Intersection of "point in plane" and "point on sphere":
  empty, point, **circle**
- Need some algebra to determine exact case, but generic case exists

Degrees-of-Freedom Analysis

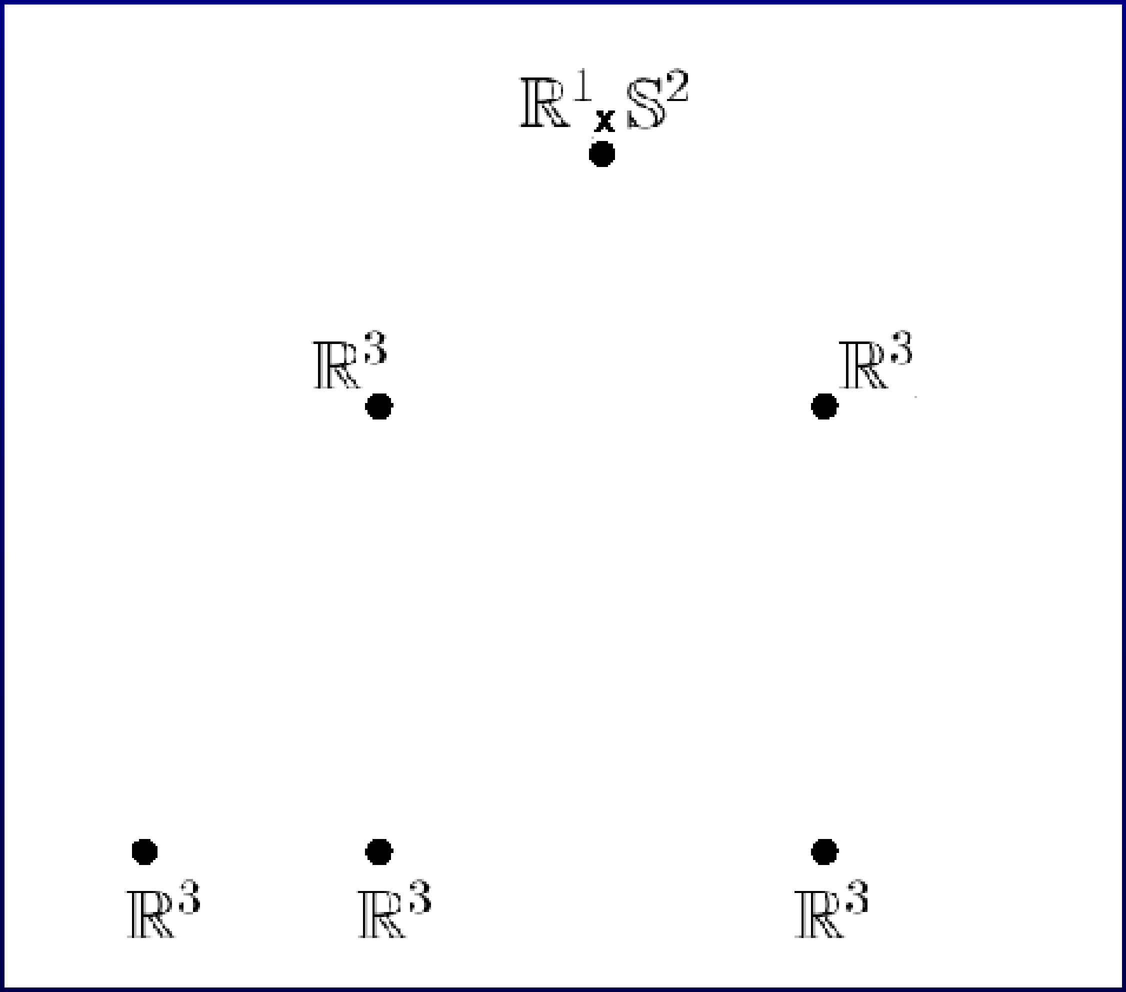
- Assume that intersections are always generic (no equations have to be solved)
- Only consider topological dimension of manifolds and sub-manifolds (locally homeomorphic manifolds)
- Constraints reduce dimension of manifolds depending on which distribution option has been chosen (local structure only)
- Find way to distribute constraints to determine structure (solvable sub-systems) of constraint system
- However, global structure of manifolds and non-generic cases are ignored

Constraint Graph

- The constraint system defines a (hyper-)graph:
  - ⋆ Geometric objects O are the vertices
  - ⋆ Types $t(o)$ are vertex labels
  - ⋆ Geometric objects involved in constraint are an edge
  - ⋆ Constraints c are edge labels
- Graph for three distances between three vertices:

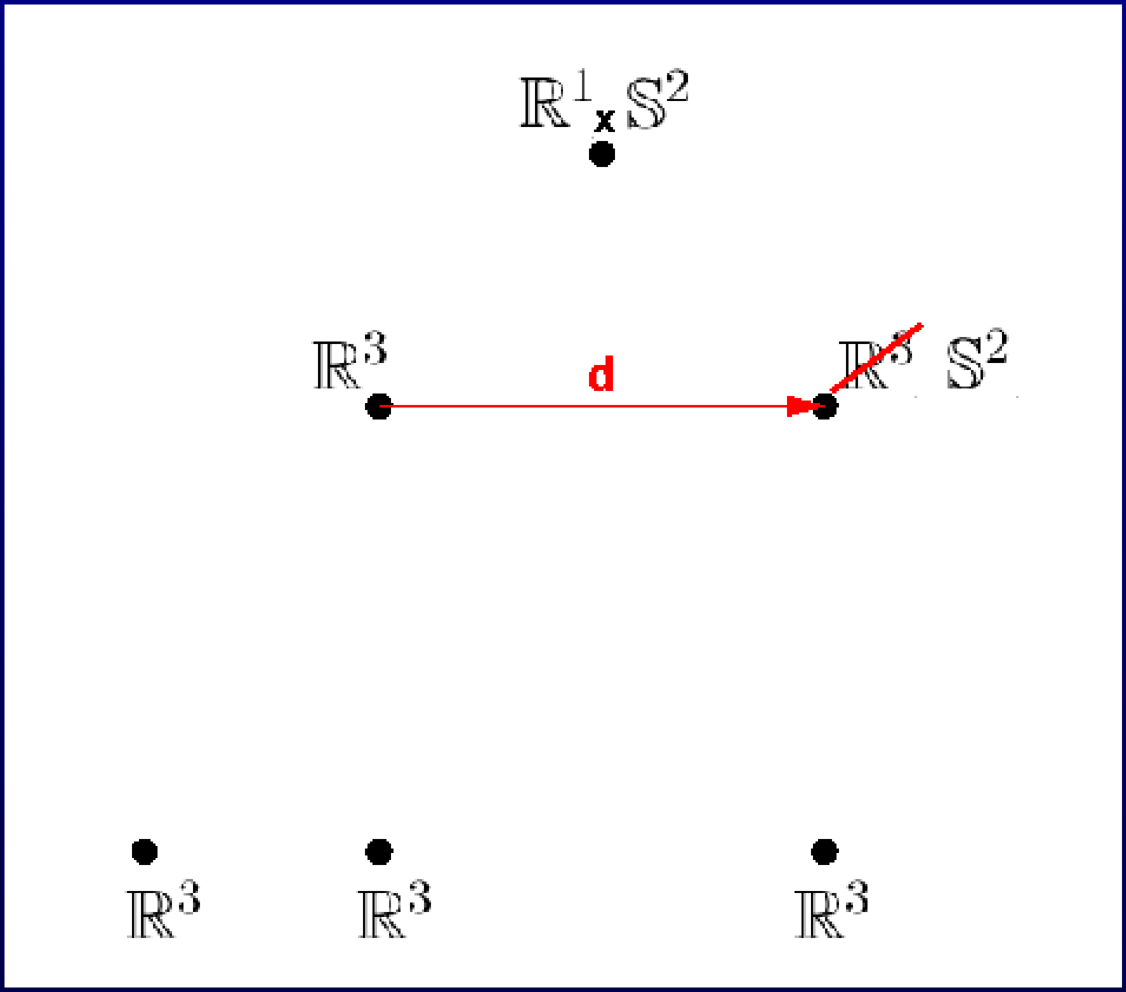


Constraint System Example



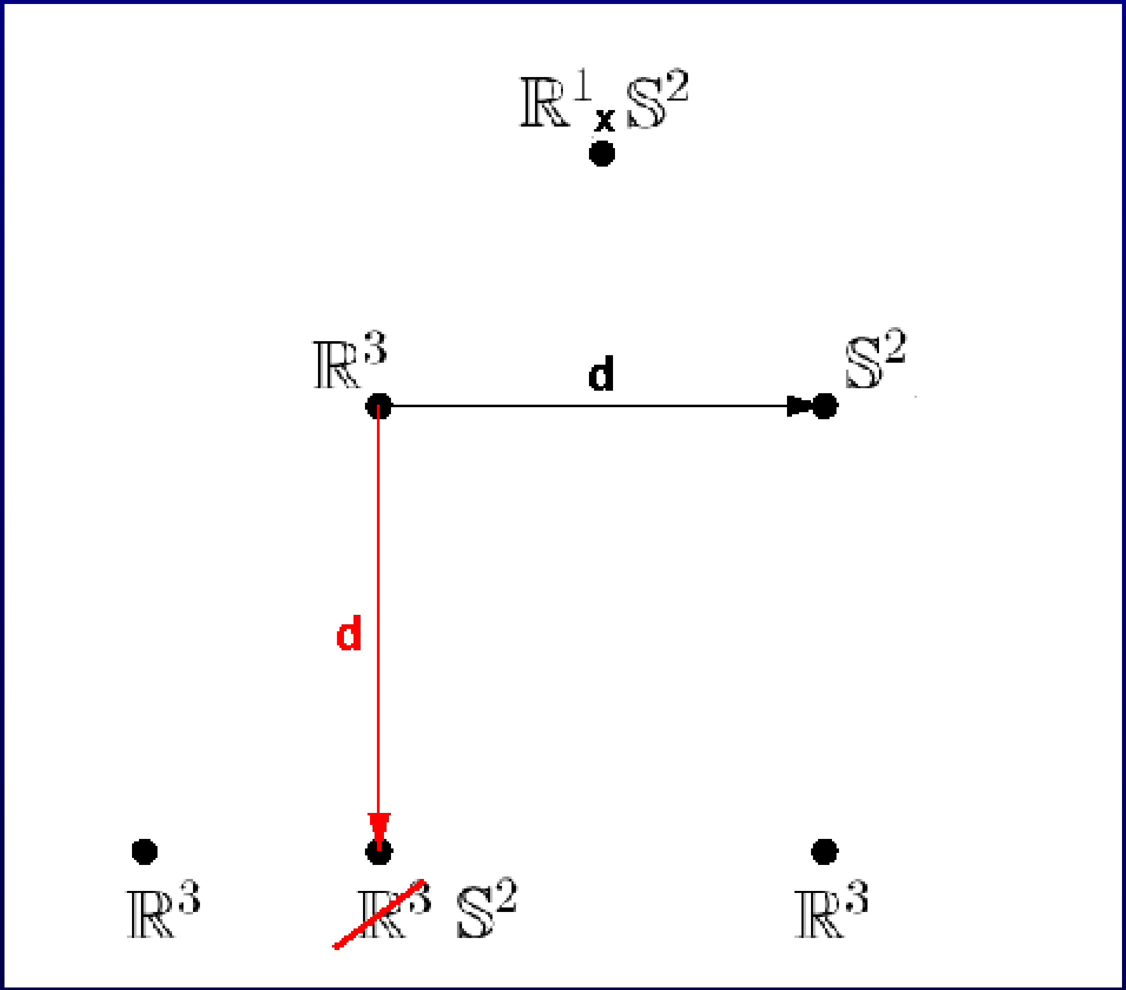
Simple example with 5 vertices, 1 plane

Constraint System Example



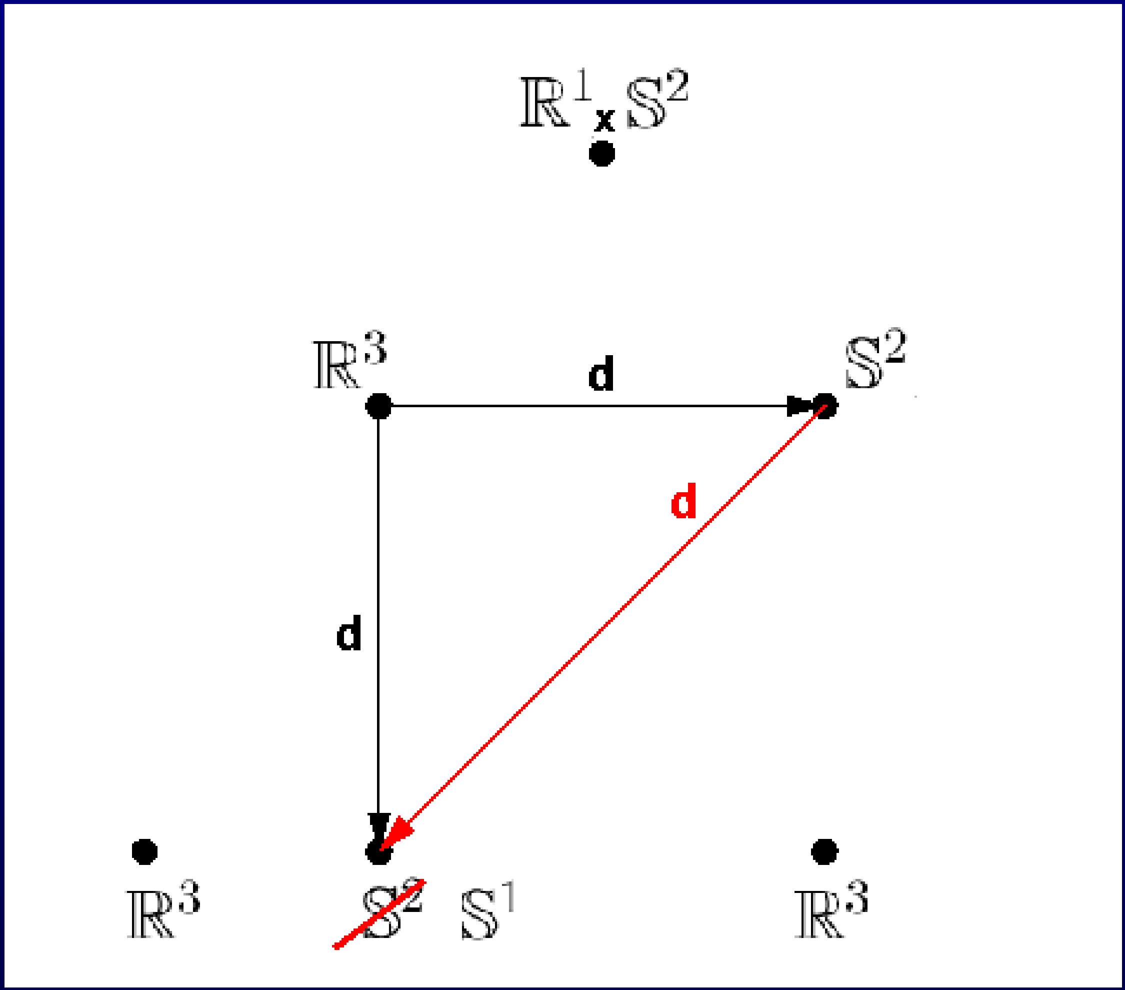
Adding distance constraint 1

Constraint System Example



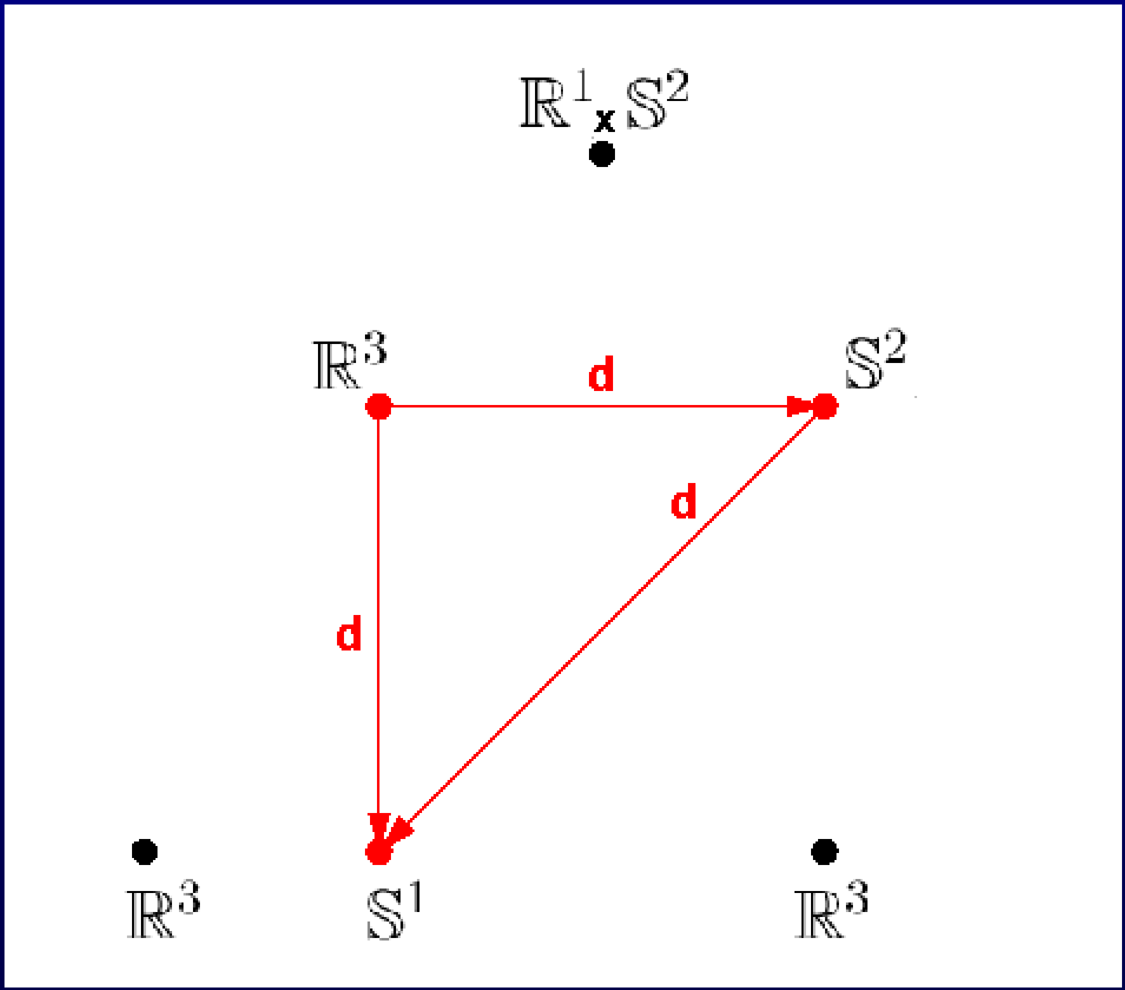
Adding distance constraint 2

Constraint System Example



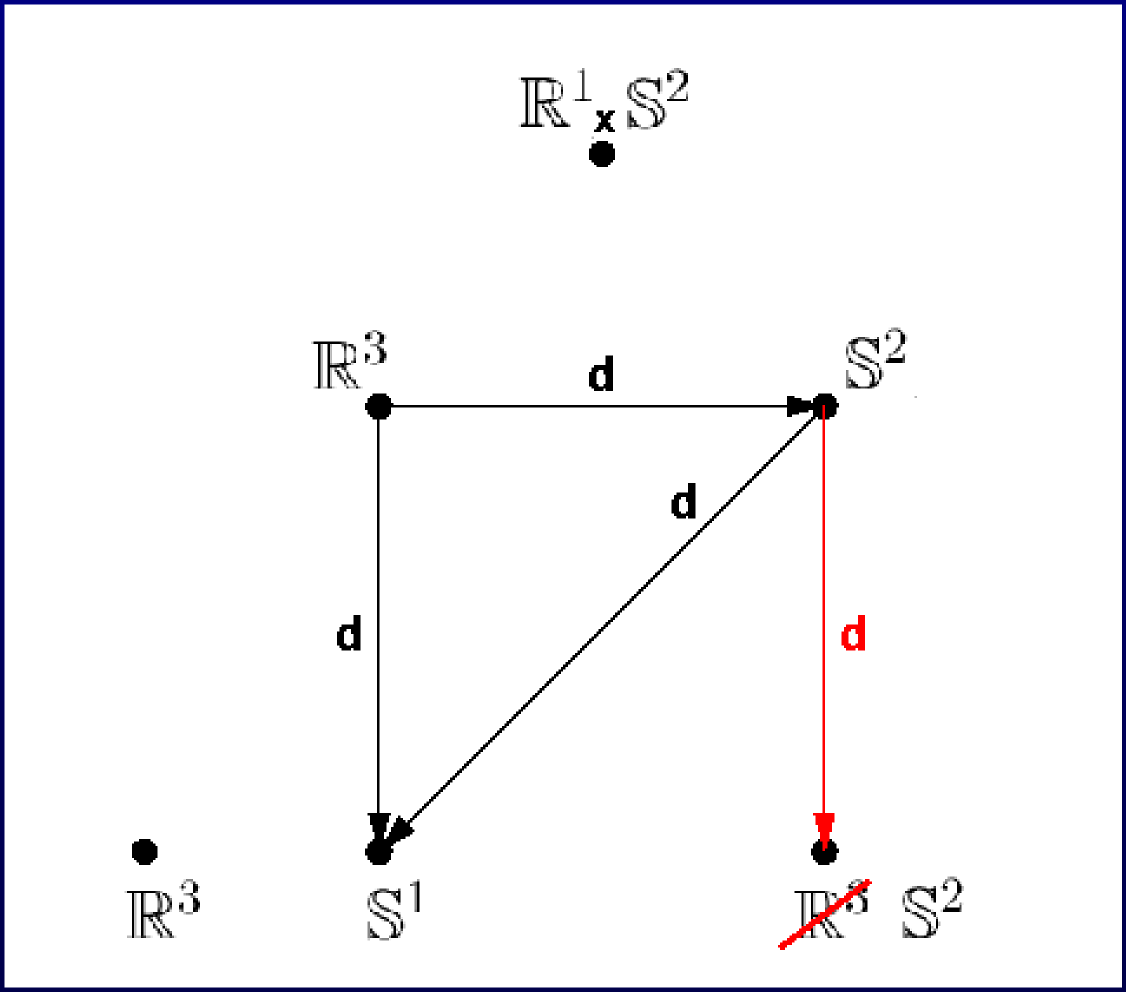
Adding distance constraint 3

Constraint System Example



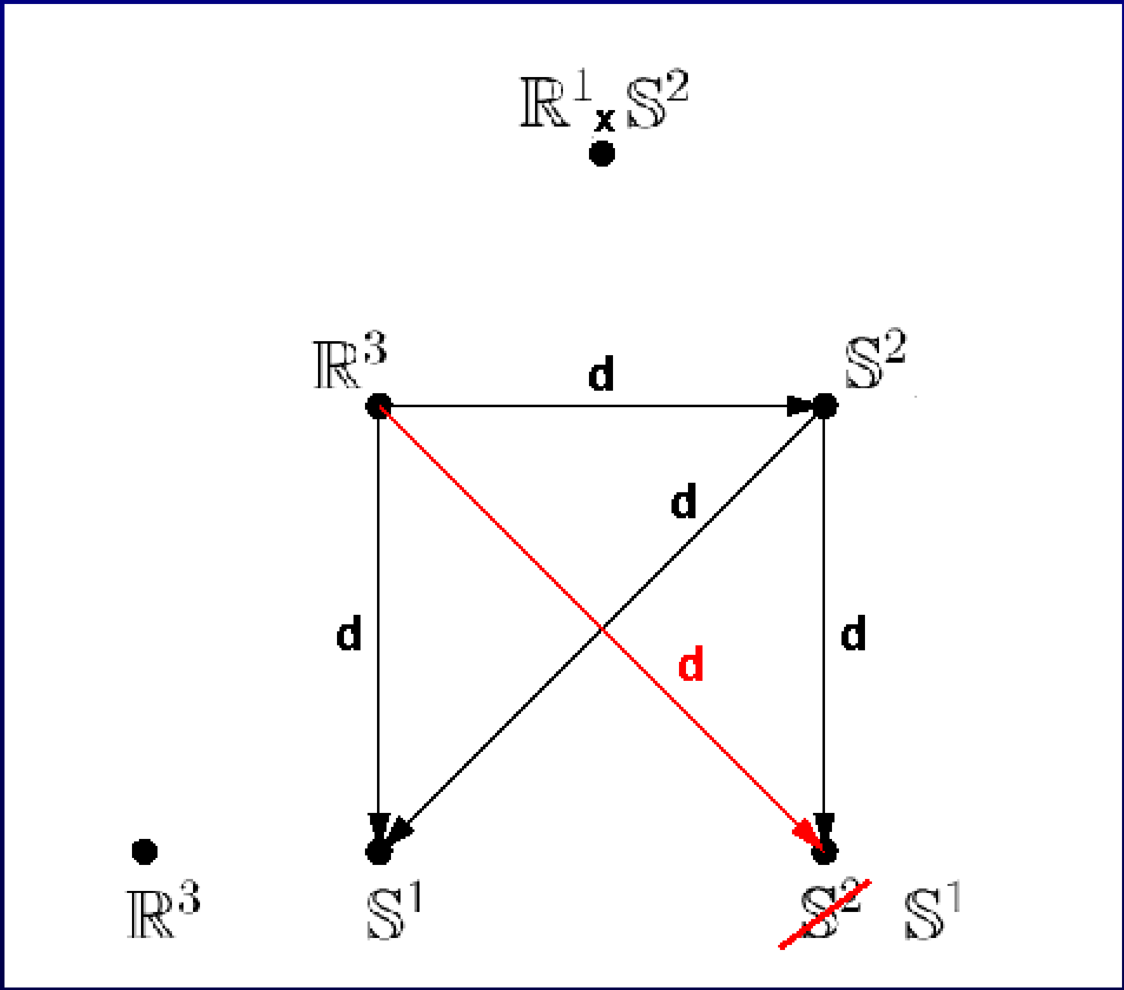
Solvable sub-system 1
$\longrightarrow$ unique modulo rotations and translations

Constraint System Example



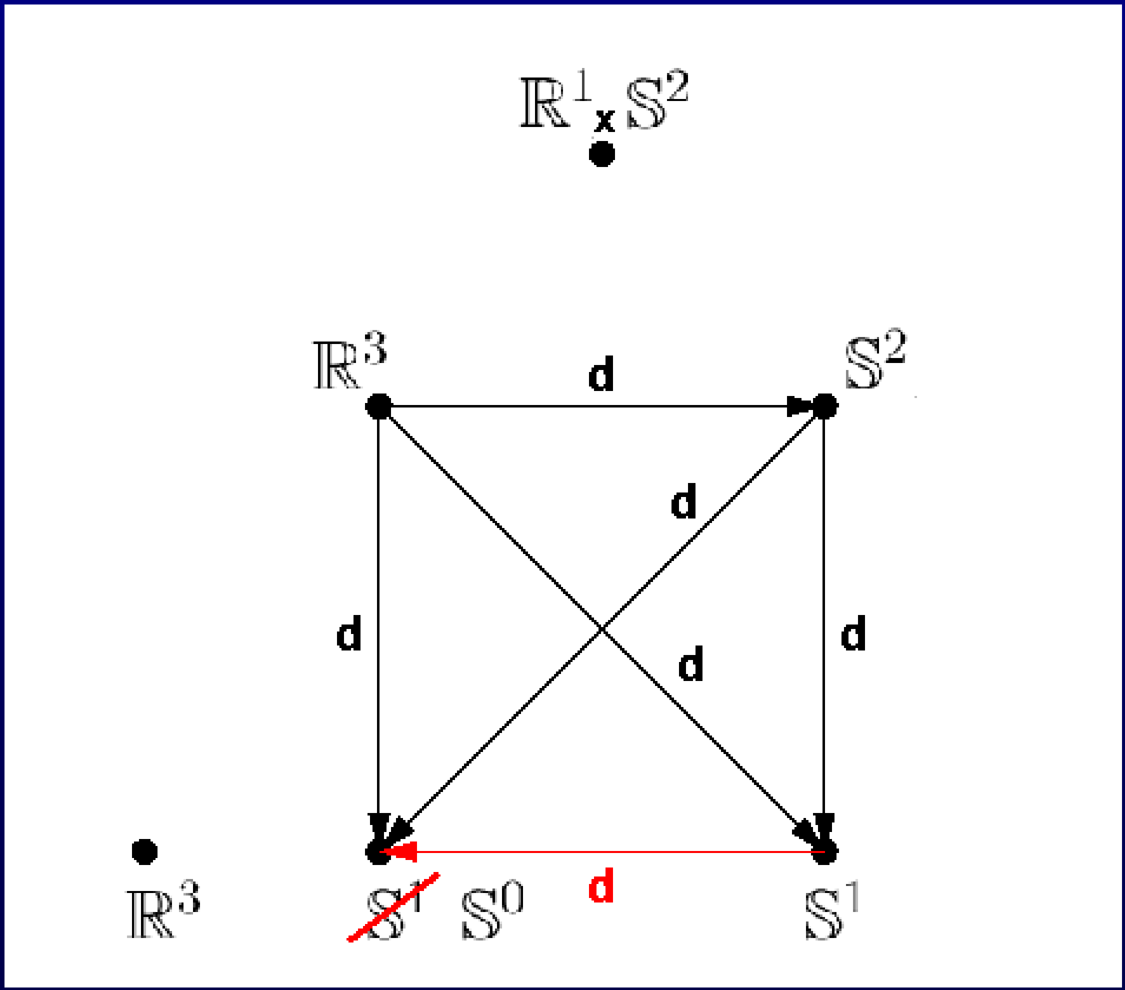
Adding distance constraint 4

Constraint System Example



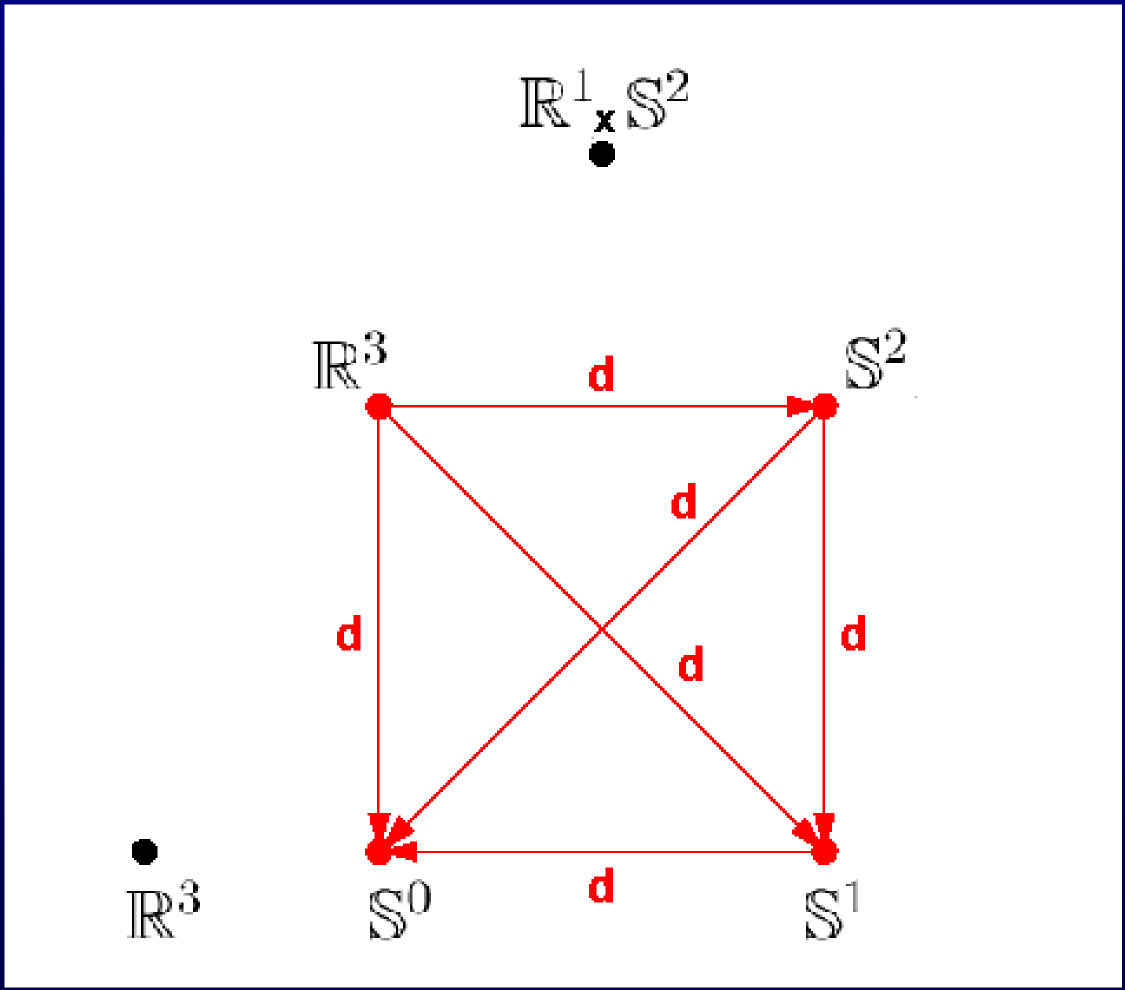
Adding distance constraint 5

Constraint System Example



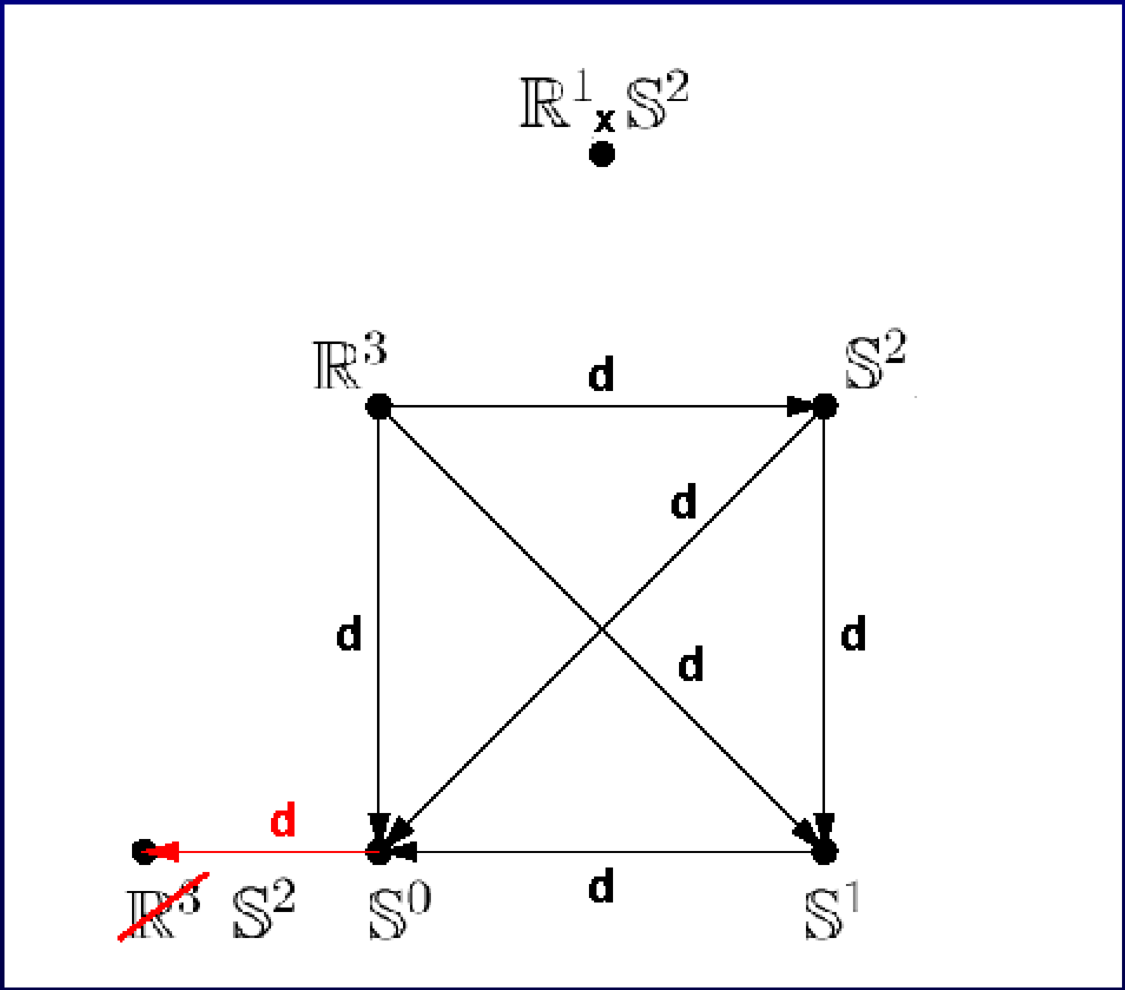
Adding distance constraint 6

Constraint System Example



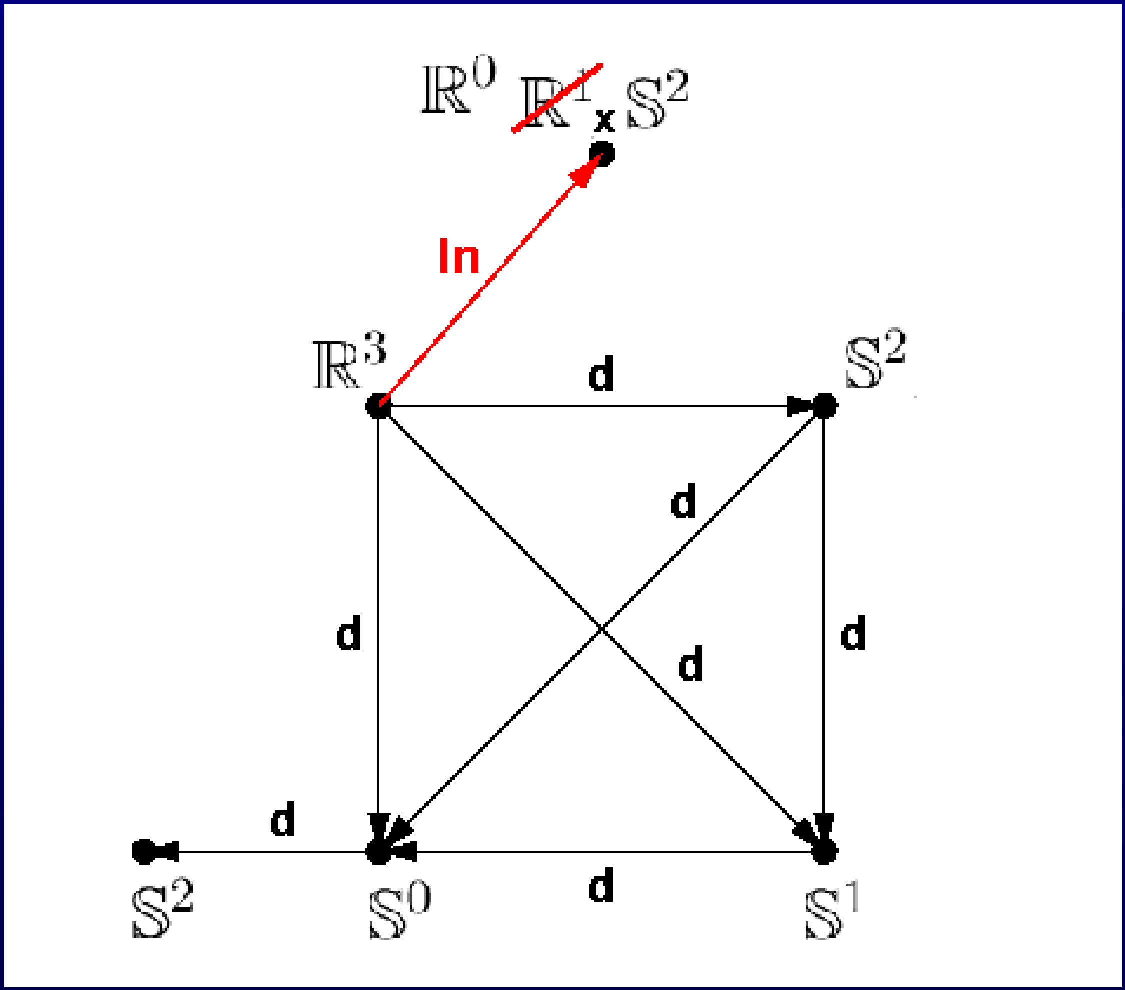
Solvable sub-system 2

Constraint System Example



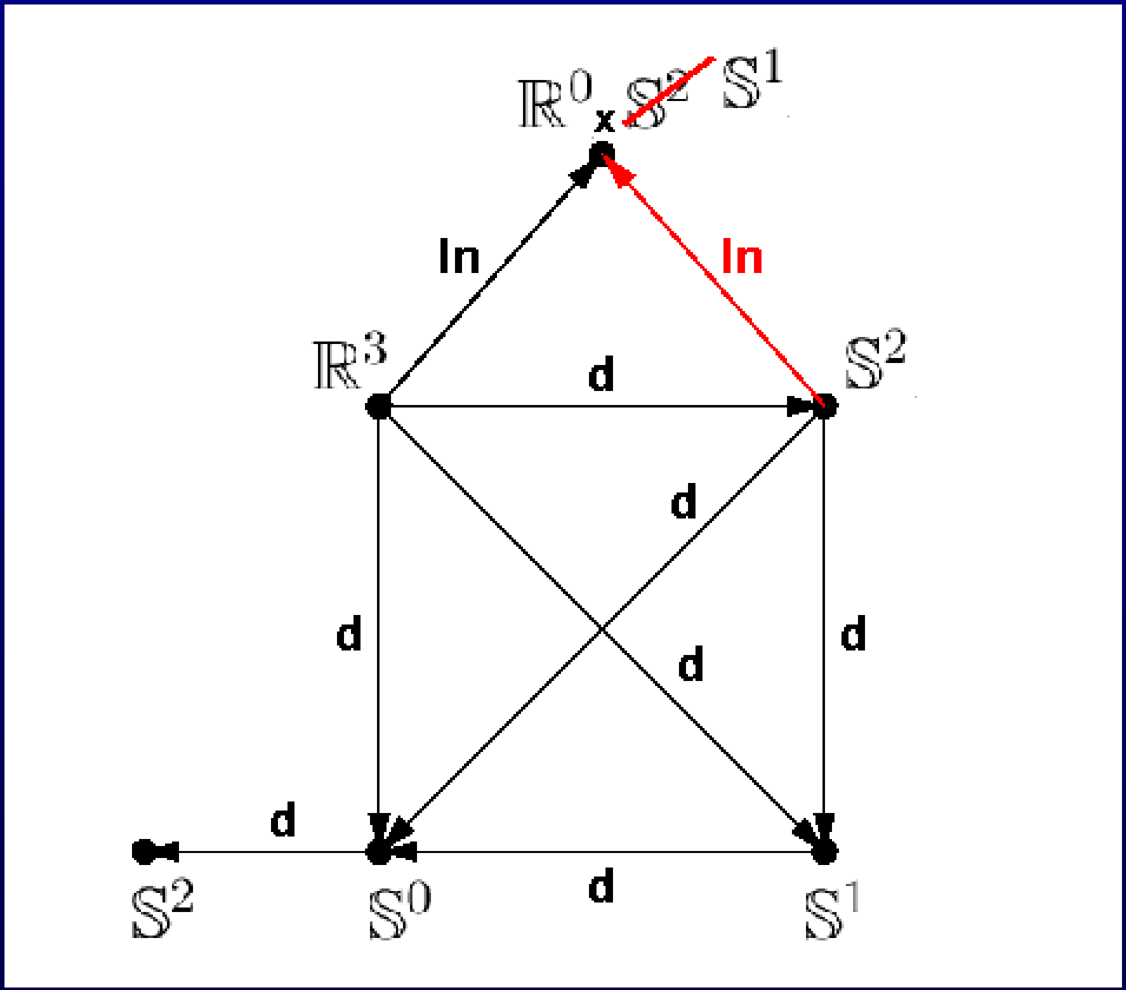
Adding distance constraint 7

Constraint System Example



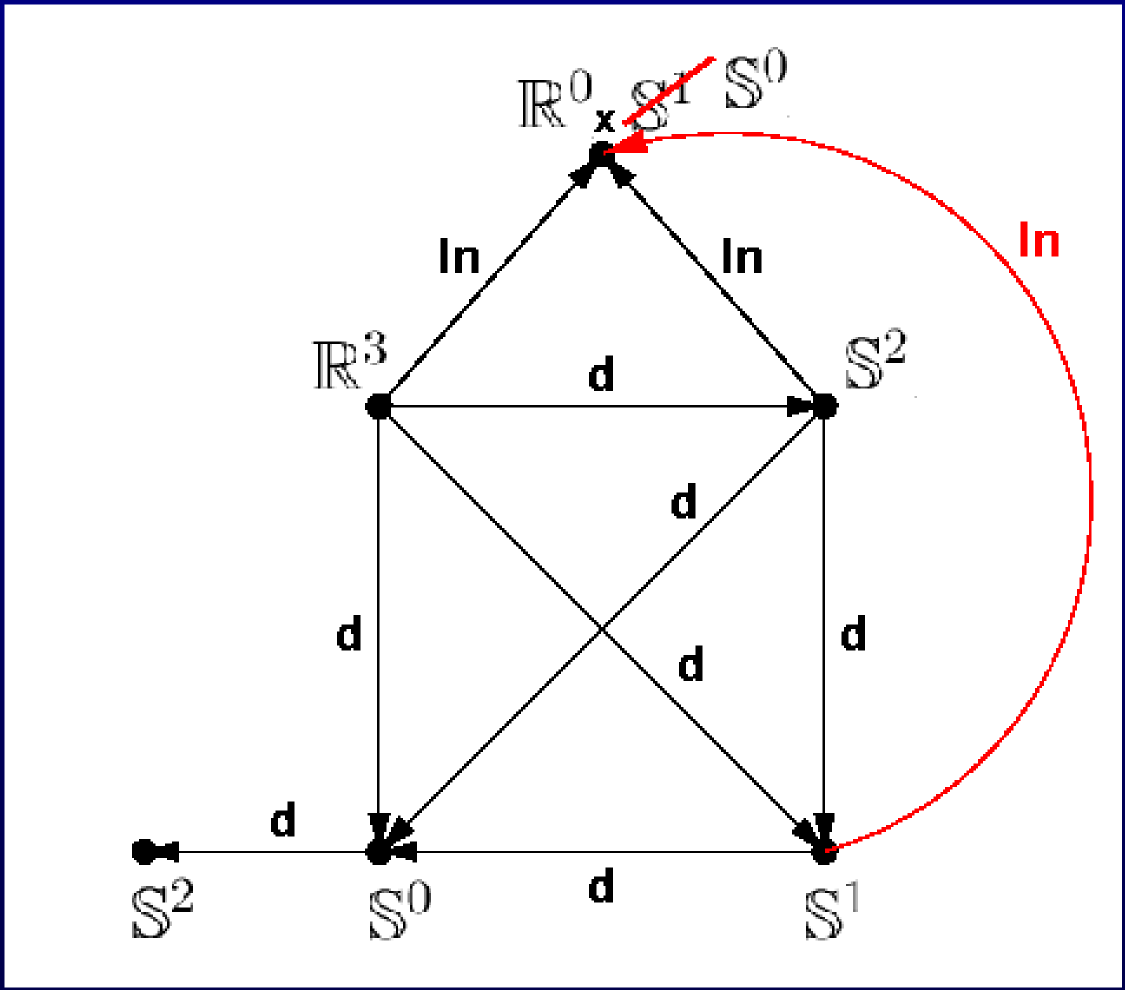
Adding vertex on plane constraint 1

Constraint System Example



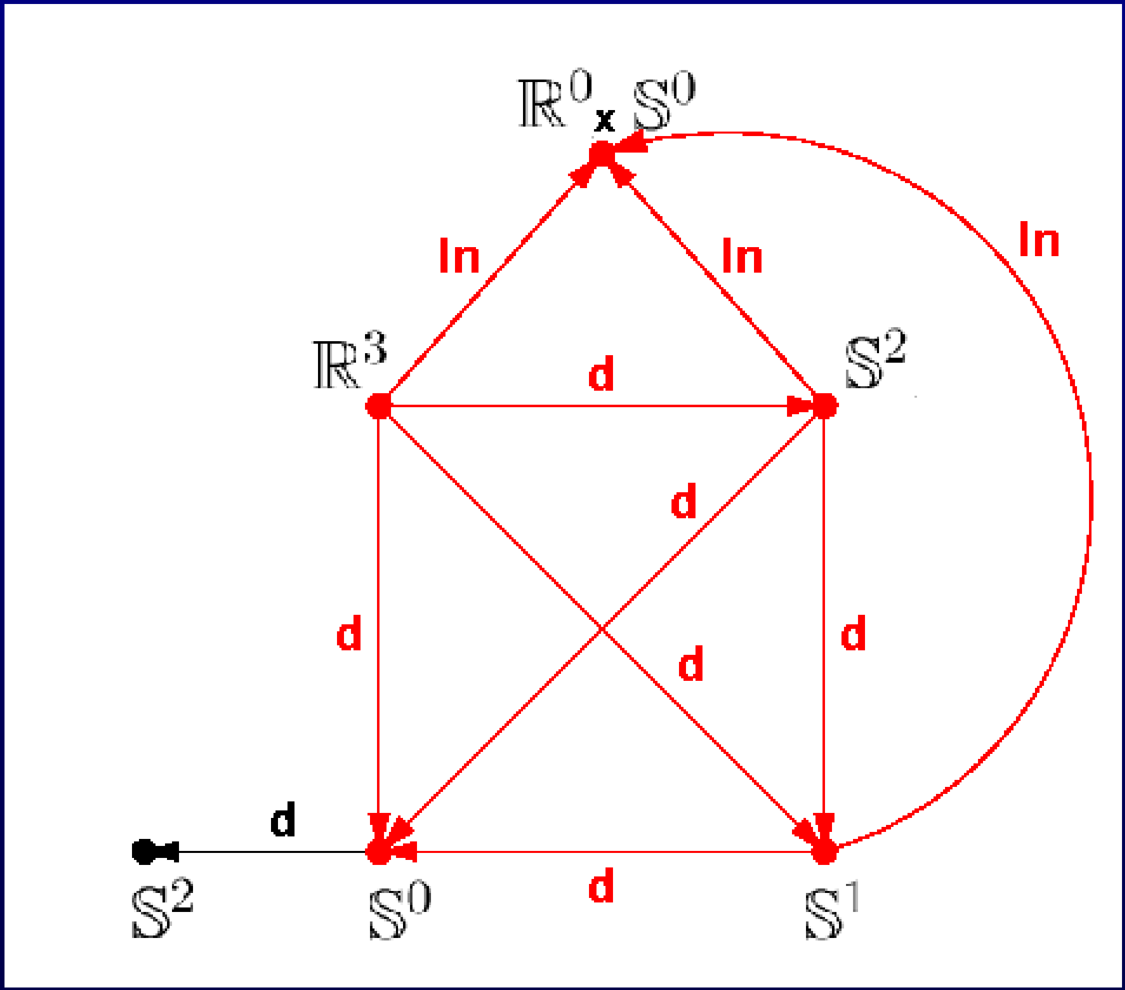
Adding vertex on plane constraint 2

Constraint System Example



Adding vertex on plane constraint 3

Constraint System Example



Solvable sub-system 3

Factoring $\mathfrak{P}(\mathbb{E}^3)$

- Geometric constraint systems usually do not specify absolute location or orientation of the shape
- Shapes (structures in $\mathbb{E}^3$ described by a geometric constraint system) which can be mapped onto each other by rotations and translations (or other transformation groups) are usually considered to be equivalent
- We are actually interested in the space of all shapes $\mathfrak{P}(\mathbb{E}^3)$ factored by rotations and translations
  (details omitted here for simplicity)

Problems of DOF Analysis

- Algebraic issues:
  - ⋆ Always assumes that intersections are generic
  - ⋆ Notion of solvable sub-system / rigid sub-structure in terms of dense sub-graphs
- Topological issues:
  - ⋆ Global structure of manifolds is ignored
  - ⋆ Sub-manifold intersections are handled only locally

Parallel Directions Constraint

- Geometric objects: directions $o_1, o_2 \in O$
$$(f(o_1), f(o_2)) \in \mathbb{S}^2 \times \mathbb{S}^2$$
- Constraint: o_1, o_2 parallel
$$(f(o_1), f(o_2)) \in \{(x_1, x_2) : x_1^t x_2 = 1\} =: c$$

⋆ c is homeomorphic to

(1) $\mathbb{S}^2 \times \mathbb{S}^0$: Choose first direction, 2nd is the same

(2) $\mathbb{S}^0 \times \mathbb{S}^2$: Analogously

⋆ But c is only locally homeomorphic to

(3) $\mathbb{S}^1 \times \mathbb{S}^1$: both objects determine part of the direction

Lines in $\mathbb{E}^3$

- We can describe a line by a position p and a direction d
- The moment vector $l = p \times d$ is independent of p
- The pair (d, l) represents normalised Plücker coordinates of a line and each six tuple fulfilling the equations below describes a line
- Manifold of all lines is 4-dimensional sub-manifold

$$\{(d, l) \in \mathbb{R}^6 : \|d\| = 1, d^t l = 0\}$$

- This is only locally homeomorphic to $\mathbb{R}^2 \times \mathbb{P}^2$ (to split positional and directional degrees of freedom of lines)

Knots

- Two topological spaces are homotopic if they can be continuously transformed into each other, e.g. a loop contracted to a point
- Consider all loops through a point x_0 on a manifold and identify homotopic loops
- This gives the fundamental group of the manifold (independent of x_0 on pathwise connected manifolds)
- Generating loops and examples (sphere, torus)

$\longrightarrow$ Gives the "knot structure" of the space and distinguishes between spaces of same dimension

Twisted Products

- Line space: $\{(d, l) \in \mathbb{R}^6 : \|d\| = 1, d^t l = 0\}$
- d is a point on a sphere with all orthogonal vectors l
- For each point on the sphere we have a plane (fibre)
- The line space is a twisted product over a sphere whose fibre is a plane

$\rightarrow$ Twisted products behave locally like a product space, but may have a different global structure