# Effect of quantum mechanical global phase factor on error versus sensitivity limitation in quantum routing

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*Abstract*— **In this paper, we explore the effect of the purely quantum mechanical global phase factor on the problem of controlling a ring-shaped quantum router to transfer its excitation from an initial spin to a specified target spin. "Quantum routing" on coherent spin networks is achieved by shaping the energy landscape with static bias control fields, which already results in the nonclassical feature of purely oscillatory closed-loop poles. However, more to the point, it is shown that the global phase factor requires a projective re-interpretation of the traditional tracking error where the wave function state is considered modulo its global phase factor. This results in a relaxation of the conflict between small tracking error and small sensitivity of the tracking error to structured uncertainties. While fundamentally quantum routing is achieved at a specific final time and hence calls for time-domain techniques, we also explore the s-domain limitations to better connect with the traditional limitations.**

## I. Introduction

We consider a spintronic network of $N$ XX-coupled spins in its single excitation subspace. The latter means that one spin and one spin only is excited, "up," while all others are "down." In this subspace, we choose a basis such that the wave function $|\Psi\rangle = e_n$, where $\{e_n\}_{n=1}^N$ is the natural basis of $\mathbb{C}^N$ over $\mathbb{C}$, denotes the quantum state where the sole excitation is on spin #$n$. The spins and couplings of such network are abstracted as vertices and edges, resp., of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. In the chosen basis, for XX-couplings, the Hamiltonian $H$ is the adjacency matrix of the graph $\mathcal{G}$ weighted by the coupling strengths with zeros on the diagonal. A simple example is given by the XX-ring structure, where the Hamiltonian has tridiagonal-like structure,

$$H = \begin{pmatrix} 0 & J_{1,2} & 0 & \dots & 0 & J_{1,N} \\ J_{2,1} & 0 & J_{2,3} & & 0 & 0 \\ 0 & J_{3,2} & 0 & & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & & 0 & J_{N-1,N} \\ J_{N,1} & 0 & 0 & \dots & J_{N,N-1} & 0 \end{pmatrix}. \tag{1}$$

In the above, $J_{m,n} = J_{n,m}$ to make the Hamiltonian Hermitian. We operate in a system of units where $\bar{h} = 1$ and the network has uniform couplings with strengths $J_{mn}$, $m \neq n$, normalized to 1.

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The "open-loop" Schrödinger equation reads $\left|\dot{\Psi}(t)\right\rangle = -\jmath H |\Psi(t)\rangle$, subject to some initial condition $\left|\Psi(0)\right\rangle = |\mathrm{IN}\rangle$, where $|\mathrm{IN}\rangle = e_i$ denotes the quantum state with the excitation on some "input" spin $i$. The control objective is to transfer the $|\mathrm{IN}\rangle$ state to some $|\mathrm{OUT}\rangle = e_o$ state where the excitation is on some "output" spin $o$. This is to be accomplished in a short amount of time $t_f$ and with maximum fidelity[1], $\mathcal{F}(t_f) := |\langle \mathrm{OUT}|\Psi(t_f)\rangle|$. This is achieved by *i-o selectively* modifying the energy landscape with static bias fields $\{D_n\}_{n=1}^N$ applied to the respective spins, resulting in the Hamiltonian $H_D = H + D$, where $D = \mathrm{diag}\{D_n\}_{n=1}^N$. The controlled Schrödinger equation becomes

$$\begin{aligned} \left|\dot{\Psi}(t)\right\rangle &= -\jmath(H+D)\,|\Psi(t)\rangle\,, \quad |\Psi(0)\rangle = |\mathrm{IN}\rangle\,, \\ &= -\jmath H\,|\Psi(t)\rangle + u(t), \quad u(t) = -\jmath D\,|\Psi(t)\rangle\,. \end{aligned} \tag{2}$$

It is observed that the right-hand side is split, somewhat artificially, into an open-loop term $-\jmath H |\Psi(t)\rangle$ and a "control" term $u(t)$. Despite the *appearance* of this control as a classical measurement-mediated feedback, it *does not* need measurement of the state (and does not create back-action of the measurements); indeed, the feedback is *field-mediated* by the physical interaction between the spins and the bias fields. Nevertheless, $u(t)$ has the *mathematical* structure of a classical feedback and as such the question is whether it is subject to some of the classical error-versus-sensitivity limitations. Classically, such limitations refer to a tracking error $|\mathrm{OUT}\rangle - |\Psi(t)\rangle$ and its sensitivity to uncertainties, but in the quantum context, the error is made smaller by considering the wave function modulo its phase factor. This paper investigates the impact of such global phase factor on the log-sensitivity of the error and points to a relaxation of the traditional conflict.

This paper follows in the footsteps of [7] with some significant differences, though. In [5], the logarithmic sensitivity was defined as $\frac{d(1-p)}{dJ}\frac{1}{1-p}$, where $p = \mathcal{F}(t_f)^2$ is the probability of successful transfer and $1-p$ interpreted as the "error," whereas in the present paper, the log-sensitivity is defined as $\frac{d(1-\mathcal{F})}{dJ}\frac{1}{1-\mathcal{F}}$. This latter relates to the projective version of the tracking error and departs more from classicality than the former logarithmic sensitivity. Moreover, in [5], the probability was averaged over a time window, while here the fidelity is instantaneous. All data is from the database [10].

[1]Sometimes the fidelity is defined as $|\langle \mathrm{OUT}|\Psi(t_f)\rangle|^2$.

## A. Notation

Throughout the paper, we consider three feedback configurations: the `CLASSICAL` configuration of Fig. 1, the $\mathcal{QUANTUM}$ configuration of Fig. 2 with the global phase factor shown in the shaded areas, and the semi-classical configuration of Fig. 2 but with the global phase factors removed. The relevant quantities are as follows:

- $\mathtt{L}(s)$, $\widehat{\mathtt{S}}(s)$, $\mathtt{T}(s) := I - \widehat{\mathtt{S}}(s)$: classical (Fig. 1) loop matrix, sensitivity and complementary sensitivity matrices, resp.
- $\mathcal{L}(s)$, $\widehat{\mathcal{S}}(s)$, $\mathcal{T}(s) = I - \mathcal{S}(s)$: projective loop matrix, sensitivity and complementary sensitivity matrices, resp., with global phase factor (shaded boxes in Fig 2). $\hat{\mathcal{S}}(s)$ is defined analytic in $\Re s > 0$.
- $\mathcal{S}(t)$: inverse Laplace transform of $\widehat{\mathcal{S}}(s)$, vanishing for $t < 0$.
- $L(s)$, $\widehat{S}(s)$, $T(s) = I - \hat{S}(s)$: loop matrix, sensitivity and complementary sensitivity matrices, resp., without global phase factor (after removal of shaded boxes in Fig 2). $\widehat{S}(s)$ is analytic in $\Re s > 0$.
- $S(t)$: inverse transform of $\hat{S}(s)$ with $S(t<0) = 0$.

# II. Classical versus quantum architecture

Laplace domain technique are of limited use in quantum control as most of the fidelity specification are rather in the time domain. Nevertheless, as shown in [6], Laplace techniques are still useful to study steady-state ($s \approx 0$) behavior. Besides, a quick review of the Laplace domain limitations are necessary to explore the classical-quantum discrepancies.

## A. `Classical`

The fundamental limitation looked at in the present paper is the quantum mechanical equivalent, if any, of $\hat{\mathtt{S}}(s) + \mathtt{T}(s) = I$, where $\hat{\mathtt{S}}(s) = (I + \mathtt{L}(s))^{-1}$ is the sensitivity matrix, $\mathtt{L}(s)$ is the loop matrix, and $\mathtt{T}(s)$ is the complementary sensitivity $\mathtt{L}(s)(I + \mathtt{L}(s))^{-1}$ of the classical loop shown in Fig. 1.

Note that the disturbance $\hat{r}(s)$ could be anything and does not support the notion of *selectivity*, that is, when $\hat{r}(s)$ is restricted to be a terminal target as shown in Fig. 2, nor does Fig. 1 support the initial conditon $|\mathrm{IN}\rangle$ of Fig. 2.

Given the classical tracking error $e(t) = r(t) - y(t)$, we have $\hat{e}(s) = \hat{\mathtt{S}}(s)\hat{r}(s)$ indicating that $\hat{\mathtt{S}}(s)$ is the transmission from the disturbance $\hat{r}(s)$ to the error $\hat{e}(s)$. $\mathtt{T}(s)$ on the other hand is related to the log-sensitivity of $\hat{\mathtt{S}}(s)$ to errors $d\mathtt{L}(s)$ in the loop matrix. Precisely,

$$\hat{\mathtt{S}}^{-1}(s) d\hat{\mathtt{S}}(s) = -((d\mathtt{L}(s))\mathtt{L}^{-1}(s))\mathtt{T}(s).$$

$\hat{\mathtt{S}}(s) + \mathtt{T}(s) = I$ hence quantifies the well known conflict between achieving simultaneously small tracking error and small log-sensitivity of tracking error to uncertainties, *disregarding selectivity.*

If $d\mathtt{L}$ is structured to represent an uncertainty on a parameter, say $J$, then the above is rewritten as

$$\frac{d \ln \hat{\mathtt{S}}(s)}{d \ln J} = -\frac{d \ln \mathtt{L}(s)}{d \ln J}\mathtt{T}(s), \tag{3}$$

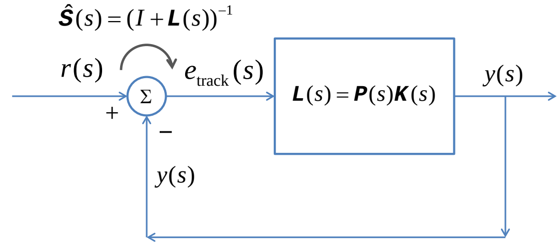


Fig. 1: `Classical` single-degree-of-freedom loop

where $d \ln \hat{\mathtt{S}} = \hat{\mathtt{S}}^{-1}(d\hat{\mathtt{S}})$ and $d \ln \mathtt{L} = (d\mathtt{L})\mathtt{L}^{-1}$. In either case, it is observed that $\mathtt{T}(s)$ is related to the log-sensitivity of $\hat{\mathtt{S}}(s)$.

## B. Phase factor and complex projective space $\mathbb{CP}^{N-1}$

In the quantum control problem of moving the system from one quantum state to another one, there is no tracking error to be minimized, but a fidelity $\mathcal{F}(t_f) = |\langle \mathrm{OUT}|\Psi(t_f)\rangle|$ to be maximized relative to $D$. However, the maximization of the fidelity can be related to minimization of a tracking error understood in some projective sense.

*Theorem 1:* The optimal controller achieving the maximum fidelity

$$\max_D |\langle \mathrm{OUT}|\Psi(t_f)\rangle|$$

is the same as the controller achieving

$$\min_D \left( \min_\phi \| \, |\mathrm{OUT}\rangle - e^{\jmath\phi} |\Psi(t_f)\rangle \, \| \right), \tag{4}$$

where $\phi$ is a global phase factor with the minimum achieved for

$$\phi^*(t_f) = -\angle\langle \mathrm{OUT}|\Psi(t_f)\rangle, \; e^{\jmath\phi^*(t_f)} = \frac{\langle \mathrm{OUT}|\Psi(t_f)\rangle^\dagger}{|\langle \mathrm{OUT}|\Psi(t_f)\rangle|}. \tag{5}$$

*Proof:* Clearly,

$$\| \, |\mathrm{OUT}\rangle - e^{\jmath\phi} |\Psi\rangle \, \|^2 = 2 - 2\Re\left(\langle \mathrm{OUT}|\Psi\rangle e^{\jmath\phi}\right).$$

Moreover,

$$\Re\left(\langle \mathrm{OUT}|\Psi\rangle e^{\jmath\phi}\right) \leq |\langle \mathrm{OUT}|\Psi\rangle|,$$

with the equality achieved when $\phi$ makes $\langle \mathrm{OUT}|\Psi\rangle e^{\jmath\phi}$ real and positive, that is, $\phi^* = -\angle\langle \mathrm{OUT}|\Psi\rangle$. It follows that

$$\| \, |\mathrm{OUT}\rangle - e^{\jmath\phi} |\Psi\rangle \, \|^2 \geq 2 - 2|\langle \mathrm{OUT}|\Psi\rangle|,$$

and

$$\min_\phi \| \, |\mathrm{OUT}\rangle - e^{\jmath\phi} |\Psi\rangle \, \|^2 = 2 - 2|\langle \mathrm{OUT}|\Psi\rangle|.$$

Finally, upon minimizing the above over all $D$'s, we obtain

$$\min_{\phi, D} \| \, |\mathrm{OUT}\rangle - e^{\jmath\phi} |\Psi\rangle \, \|^2 = 2 - 2\max_D |\langle \mathrm{OUT}|\Psi\rangle|,$$

and the theorem is proved. ∎

The preceding theorem states that controllers can as well be optimized (although in a somewhat computationally clumsy way) on the basis of the *projective tracking error*

$$e_{\text{proj}}(t) = |\text{OUT}\rangle - e^{\jmath\phi^*(t)}\,|\Psi(t)\rangle \qquad (6)$$

with the already perceived reward that the above connects with classical concepts.

More formally speaking, since $\|\,|\Psi\rangle\,\|_{\mathbb{C}^N} = 1$ and since a phase factor $\exp(-\jmath\phi)$ does not fundamentally change the quantum state, $|\Psi\rangle$ lives in $\mathbb{S}^{2N-1}/\mathbb{S}^1 = \mathbb{CP}^{N-1}$, the complex projective space. Observe that the fidelity $|\langle \text{OUT}|\Psi(t_f)\rangle|$ is the cosine of the Fubini-Study metric on $\mathbb{CP}^{N-1}$. More closely related to (6), observe the following:

*Corollary 1:* $\delta(|\text{OUT}\rangle, |\Psi\rangle) := \min_\phi \|\,|\text{OUT}\rangle - e^{-\jmath\phi}\,|\Psi\rangle\,\|_{\mathbb{C}^N}$ is a metric on $\mathbb{CP}^{N-1}$.

*Remark 1:* The global phase $\phi^*(t)$ could be viewed as an ad hoc trick to think maximum fidelity in terms of $\delta$-minimum tracking error. However, for it to have its classical quantum mechanical interpretation, it needs to be constant, which could be accomplished by limiting it to $\phi^*(t_f)$. However, a time-varying global phase $\phi^*(t)$ could have the quantum mechanical interpretation of change of the zero energy level. Indeed, a shift of energy level $H_D \to H_D + cI$ yields a phase factor $\exp(-\jmath ct)$. From (5), under near perfect state transfer, it follows that this specific global phase factor could be associated with a shift $c = \langle \text{OUT}|H_D|\text{IN}\rangle$. ■

### C. Projective sensitivity

Observing from (2) that $|\Psi(t)\rangle = e^{-\jmath H_D t}\,|\text{IN}\rangle$ and defining the output-input swapping operator

$$P = |\text{IN}\rangle\,\langle \text{OUT}|$$

the projective tracking error leads to the concept of *projective sensitivity function* $\mathcal{S}(t)$,

$$e_{\text{proj}}(t) = \underbrace{\left(I - e^{\jmath\phi^\star(t)}e^{-\jmath H_D t}P\right)}_{\mathcal{S}(t)}|\text{OUT}\rangle\,. \qquad (7)$$

The connection with the `classical` relationship $\hat{e}(s) = \hat{\mathsf{S}}(s)\hat{r}(s)$ is obvious, but note the selectivity feature of the above that the disturbance $|\text{OUT}\rangle$ is selectively restricted to be a natural basis of $\mathbb{C}^N$. In fact, the selectivity is 2-fold, as contrary to a `classical` controller, $D$ is not universal, as it is selectively optimized for $|\text{OUT}\rangle$. To connect the above with the fidelity, observe that

$$\begin{aligned}\langle \text{OUT}|\mathcal{S}(t)|\text{OUT}\rangle &= \langle \text{OUT}\,|e_{\text{proj}}(t)\rangle \\ &= 1 - \left(\langle \text{OUT}|e^{-\jmath H_D t}\,|\text{IN}\rangle\right)e^{\jmath\phi^*(t)} \\ &= 1 - \mathcal{F}(t), \qquad (8)\end{aligned}$$

where the third equality is seen by remembering that $\phi^*(t)$ is chosen so as to make $\left(\langle \text{OUT}|e^{-\jmath H_D t}\,|\text{IN}\rangle\right)e^{\jmath\phi^*(t)}$ real and positive.

Here we are at the crucial point. Even though quantum transport is usually formulated in terms of fidelity, Eq. (8) reveals that we could equally argue in terms of the *projective sensitivity function* $\mathcal{S}(t)$, or more specifically in terms of the *selective* projective sensitivity $\langle \text{OUT}|\mathcal{S}(s)\,|\text{OUT}\rangle$. Moreover, sensitivity of the fidelity can be argued in terms of the sensitivity of the *selective* sensitivity function $\langle \text{OUT}|\mathcal{S}(t)|\text{OUT}\rangle$.

Fidelity is usually formulated as above in the time-domain; however, Laplace domain techniques have also been used [6] but in the very specific context of steady-state behavior ($s \approx 0$). Nevertheless, to better connect with the classical concepts, usually formulated in the Laplace domain, we define

$$\hat{e}_{\text{proj}}(s) = \underbrace{\left(I/s - \widehat{e^{\jmath\phi^\star(t)}} \star (sI + \jmath H_D)^{-1}P\right)}_{\hat{\mathcal{S}}(s)}|\text{OUT}\rangle\,, \quad (9)$$

where the widehat notation denotes the unilateral Laplace transform and $\star$ denotes the Laplace domain convolution

$$(\hat{X} \star \hat{Y})(s) = \frac{1}{2\pi\jmath}\int_{c-\jmath\infty}^{c+\jmath\infty} \hat{X}(s-z)\hat{Y}(z)dz, \qquad (10)$$

where the integration path is a vertical line in the common $z$-domain of convergence of $\hat{X}(s-z)$ and $\hat{Y}(z)$, assuming such a nonempty intersection exists. Details are relegated to Appendices A, B, C, and D.

The problem is that $\hat{\mathcal{S}}(s)$ does not naturally lend itself to a representation of the form $(I + \mathcal{L}(s))^{-1}$ with the idea that $\mathcal{L}(s)$ factors as $\mathcal{P}(s)\mathcal{K}(s)$, where $\mathcal{P}(s)$ is some plant and $\mathcal{K}(s)$ some controller. Formally, we could define a fictitious loop matrix $\mathcal{L} = (I + \hat{\mathcal{S}})^{-1}$, but it is unlikely that it would factor as $\mathcal{PK}$. At best, $\hat{\mathcal{S}}(s)$ can be related to the architecture shown in Fig. 2, which is certainly not of the single degree of freedom configuration, but could be interpreted as a 3-degree of freedom one, notwithstanding the feedbacks involved in the phase function.

Following the classical path of ideas, we define a fictitious loop matrix $\mathcal{L}$ to reproduce the classical relation $\hat{\mathcal{S}}(s) = (I + \mathcal{L})^{-1}(s)$, that is, $\mathcal{L}(s) = \hat{\mathcal{S}}^{-1}(s) - I$; explicitly

$$\mathcal{L}(s) = \left(I/s - \widehat{e^{\jmath\phi^\star(t)}} \star (sI + \jmath H_D)^{-1}P\right)^{-1} - I.$$

Using the matrix inversion lemma, the projective loop matrix can be rewritten as

$$\begin{aligned}\mathcal{L}(s) =&(s-1)I \\ &+ s^2\left(\left(\widehat{e^{\jmath\phi^\star(t)}} \star (sI + \jmath H_D)^{-1}\right)^{-1} - Ps\right)^{-1}P.\end{aligned}$$

### D. Classical oscillatory systems

Schrödinger's equation (2) is, after all, an Ordinary Differential Equation (ODE) over $\mathbb{C}^n$ and should the eigenvalues of $\jmath H_D$ come in complex conjugate pairs, it could be interpreted as a lossless spring mechanical system or a LC oscillatory circuit. Moreover, "energy landscape" techniques have been popular in robotics and electromechanical systems [8], [11], where the energy is shaped so as to put its minimum at the target by *local* feedbacks bearing similarity with $u_k = -\jmath D_k \Psi_k$. Such classical systems follow the architecture of Fig. 2—without the global phase factors in the shaded

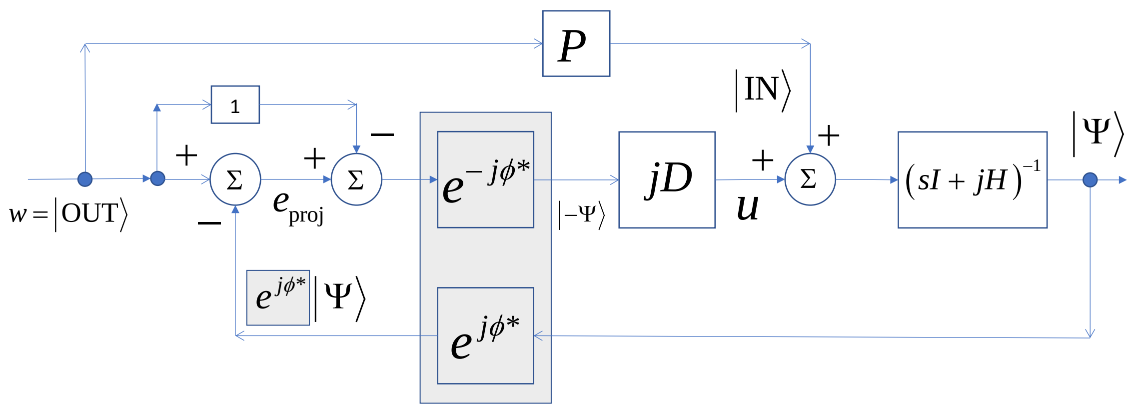


Fig. 2: The projective error $e_{\text{proj}}$ embedded in a semi-classical 3-degree-of-freedom loop. The top paths ($P$ and 1) are indeed two additional degrees of freedom relative to the single degree of freedom configuration. The shaded areas refer to the "global phase factor." Note that the $e^{\pm \jmath\phi^*}$ operations have to be interpreted in the time-domain.

areas—with relevant tracking error defined as, reverting to classical notation,

$$e(t) = \underbrace{\left(I - e^{-\jmath H_D t} P\right)}_{S(t)} |\text{OUT}\rangle\,, \tag{11}$$

or taking the unilateral Laplace transform

$$\hat{e}(s) = \underbrace{\left(I/s - (sI + \jmath H_D)^{-1} P\right)}_{\widehat{S}(s)} |\text{OUT}\rangle\,, \tag{12}$$

together with the fictitious loop function

$$L(s) = (s-1)I + s^2 \left(s(I-P) + \jmath H_D\right)^{-1} P.$$

The importance of this case-study is that comparison between the two sensitivity matrices $\widehat{S}(s)$ and $\hat{S}(s)$ would narrow down quantum enhancement, if any, regarding circumventing the classical limitations. This is essentially what is addressed in Sec. V-C.

## III. Sensitivity—Laplace domain

### A. Selective sensitivity

Taking the Laplace transform of (8) and using (9) yields

$$\langle \text{OUT}|\hat{\mathcal{S}}(s)|\text{OUT}\rangle = 1/s - \hat{\mathcal{F}}(s).$$

Taking the log-differential, while remembering that nominally $J_{mn} = 1$, yields

$$\frac{d\langle \text{OUT}|\hat{\mathcal{S}}(s)|\text{OUT}\rangle}{dJ_{mn}} \frac{1}{\langle \text{OUT}|\hat{\mathcal{S}}(s)|\text{OUT}\rangle} \tag{13}$$

$$= -\left\langle \text{OUT} \left| \frac{d\hat{\mathcal{F}}(s)}{dJ_{mn}} \right| \text{OUT} \right\rangle \frac{1}{1/s - \hat{\mathcal{F}}(s)}, \tag{14}$$

where the computations of $\hat{\mathcal{S}}(s)$ and $d\hat{\mathcal{S}}(s)/dJ_{mn}$ are expanded upon in Appendices B, C, and D. Such quantities are numerically explored in Sec. V-C.

### B. Motivation for Laplace techniques: Asymptotic results

Here we provide motivation for the sensitivity analysis of $\hat{\mathcal{S}}(s)$. We proceed from the explicit expressions for $\hat{\mathcal{S}}(s)$ and $d\hat{\mathcal{S}}(s)$ of Appendix D and use a generalized Laplace final value theorem to derive some asymptotic behavior of $\mathcal{S}(t)$, $d\mathcal{S}(t)$ as $t \to \infty$. Moreover, in the quest for a quantum enhancement, we contrast those results with the limiting behavior of $S(t), dS(t)$ when they do not include the global phase factor (no shaded boxes in Fig. 2).

Since our systems are not closed-loop stable in the classical sense, we need a generalized Laplace final value theorem [2]:

*Theorem 2: Nonclassical* Laplace final value theorem [4, Th. 2]. Let $\hat{f}(s)$ be the Laplace transform of $f(t)$. If

$$\lim_{s\to 0} \int_s^\infty \frac{\hat{f}(\xi)}{\xi} d\xi = \infty,$$

then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\tau) d\tau = \lim_{s\to 0} s\hat{f}(s). \qquad \blacksquare$$

In the following, we highlight the difference between the two cases: with and without global phase factor (with and without shaded boxes in Fig. 2) as a way to gauge quantum effects.

*Theorem 3:* Regarding the *average* steady-state error in the sense of Th. 3, we have

1) *With global phase factor:*

$$\lim_{s\downarrow 0} \langle \text{OUT}|s\hat{\mathcal{S}}(s)|\text{OUT}\rangle = 1 - \sum_k |\langle \text{OUT}|\Pi_k|\text{IN}\rangle|^2.$$

[2] Ph. Anderson in his famous localization paper [1] was aware of and utilized this result, but not with the level of rigor as in [4].

2) *Without global phase factor:*

$$\lim_{s\downarrow 0}\langle \text{OUT}|s\hat{S}(s)|\text{OUT}\rangle = 1 - \langle \text{OUT}|\text{IN}\rangle, \\ = \begin{cases} 1 \text{ for transfer,} \\ 0 \text{ for localization.} \end{cases}$$

*Proof:* With global phase factor, we need to chase the $1/s$ term in (19), which is easily accomplished by setting $k = \ell$. To remove the global phase factor, just remove $e_i^\dagger \Pi_k e_0$ in (19), and remember that $\sum_\ell \Pi_\ell = I$. ■

*Theorem 4:* For the differential $\langle \text{OUT}|sd\hat{\mathcal{S}}(s)|\text{OUT}\rangle$, we have

1) *With global phase factor:*

$$\lim_{s\downarrow 0}\langle \text{OUT}|sd\hat{\mathcal{S}}(s)|\text{OUT}\rangle = \begin{cases} \infty \text{ for transfer,} \\ 0 \text{ for localization.} \end{cases}$$

2) *Without global phase factor and with $\lambda_k(H_D) \neq 0$:*

$$\lim_{s\downarrow 0}\langle \text{OUT}|sd\hat{\mathcal{S}}(s)|\text{OUT}\rangle = 0.$$

*Proof:* With global phase factor, for finiteness of $\lim_{s\downarrow 0} sd\hat{\mathcal{S}}(s)$ we look at the $1/s^2$ term in (20), which is found by setting $k = \ell = m$,

$$2\Im \sum_k \left( (e_i^\dagger \Pi_k dH_D \Pi_k e_o)(e_o^\dagger \Pi_k e_i) \right).$$

Taking the localization case, $e_i = e_0$, the above is easily seen to vanish as $\Pi_k$ and $dH_D$ are all Hermitian operators. Hence, finiteness for localization. For transfer, the above does not vanish; hence, $\lim_{s\downarrow 0}$ is unbounded. The result without global phase factor is easily derived from (12). ■

### C. Comparison with classical, nonselective sensitivity

Because the relationship between $\mathcal{L}$ and $\hat{\mathcal{S}}$ is the same as that between $\mathtt{\hat{L}}(s)$ and $\mathtt{S}(s)$, the log-sensitivity of $\hat{\mathcal{S}}(s)$ with respect to coupling parameters in $\mathcal{L}$ is structurally the same as (3),

$$\frac{d\ln \hat{\mathcal{S}}(s)}{d\ln J_{mn}} = -\frac{d\ln \mathcal{L}(s)}{d\ln J_{mn}}\mathcal{T}(s), \tag{15}$$

where $\mathcal{T}(s) = \mathcal{L}(s)(I+\mathcal{L}(s))^{-1}$. The above comes together with the obvious relationship

$$\hat{\mathcal{S}}(s) + \mathcal{T}(s) = I.$$

The above might be called the quantum mechanical error versus sensitivity limitation, with the caveat that it does not support the selectivity of the quantum transport.

Nevertheless, disregarding $|\text{IN}\rangle, |\text{OUT}\rangle$ selectivity, we could look at structured uncertainties and we could claim a *quantum robustness enhancement* when

$$\sigma_{\max}\left(\frac{d\ln \mathcal{L}(s)}{d\ln J_{mn}}\right) \ll \sigma_{\max}\left(\frac{d\ln L(s)}{d\ln J_{mn}}\right)$$

for the usual $\sigma_{\max}$ or any other matrix-norm for that matter.

Instead of arguing on the sensitivity relative to some artificially defined loop matrix, we might as well look at the log-sensitivity relative to physical embedded in $H_D$, both the physical coupling parameters as well as the control parameters $D_n$. Quantum enhancement relative to coupling and control errors is claimed if

$$\frac{d\ln \hat{\mathcal{S}}(s)}{dJ_{mn}} \ll \frac{d\ln \hat{S}(s)}{dJ_{mn}}.$$

## IV. Sensitivity—Time domain

The starting point of the time-domain analysis is the sensitivity of the matrix exponential to variation in the matrix exponent, as given by the Zassenhaus formula [3]:

$$\begin{aligned} \exp(-\jmath(H_D+dH_D)t) = \\ &\exp(-\jmath H_D t)\exp(-\jmath dH_D t) \\ &\times \Pi_{p=2}^{\infty} \exp(Z_p(H_D, dH_D)(-\jmath t)^p), \end{aligned}$$

where $Z_p(\cdot,\cdot)$ is a homogeneous Lie polynomial of degree $p$, and the decomposition is unique. Note that $Z_p(H_D, dH_D)$ contains a linear term in $dH_D$, which should be taken into consideration when computing sensitivity. Explicitly,

$$\begin{aligned} e^{-\jmath(H_D+dH_D)t} =& e^{-\jmath H_D t} e^{-\jmath dH_D t} e^{\frac{1}{2}[H_D, dH_D]t^2} \\ &\times e^{\frac{\jmath t^3}{6}[[H_D,dH_D],H_D]} \\ &\times e^{-\frac{t^4}{24}[[[H_D,dH_D],H_D],H_D]} \cdots \end{aligned}$$

Setting $dH_D = dJ_{mn}S_{mn}$, where $dJ_{mn}$ is the variation of the parameter $J_{mn}$ and $S_{mn}$ the associated structure and utilizing the above formula with its expansion restricted to include polynomials up to $Z_2$ yields

$$\frac{de^{-\jmath H_D t}}{dJ_{mn}} \approx e^{-\jmath H_D t}\left(-\jmath S_{mn}t + \frac{1}{2}[H_D, S_{mn}]t^2\right). \tag{16}$$

While approximate, this formula has the merit that it reveals the role of the commutator $[H_D, S_{mn}]$.

From (8), the time-domain log-sensitivity is set up as

$$\frac{d(1-\mathcal{F})}{dJ_{mn}}\frac{1}{1-\mathcal{F}} = -\frac{\left\langle \text{OUT}\left|\frac{d\mathcal{S}(t)}{dJ_{mn}}\right|\text{OUT}\right\rangle}{\langle \text{OUT}|\mathcal{S}(t)|\text{OUT}\rangle}, \tag{17}$$

where $d\mathcal{S}(t)/dJ_{mn}$ is computed from Eqs. (7), (16), and $e^{\jmath\phi^*(t)}$ is evaluated as given by Th. 1. Note that for numerical computations, Eq. (16) might not be accurate enough, in which case we have to revert to [13, Eq. 32]. The details are left out.

## V. Numerical results

### A. Jonckheere-Terpstra statistic

The fundamental question is whether the plots of error and log-sensitivity versus controller are concordant or discordant. This is a traditional nonparametric rank correlation issue that started with von Neumann [15], [16], and further developed by Kendall [9], Jonckheere [5] and Terpstra [14]. Here we order the controllers by increasing error and assess whether the log-sensitivity is increasing using the Jonckheere-Terpstra (JT) test. Such increase would be an anti-classical trend, as classically one would expect conflict between error and its log-sensitivity. The JT test partitions the log-sensitivity data

$Y$ into $I$ (here $I = 5$) bins; it computes the median $\tilde{Y}_i$ of each bin, and set the null hypothesis $H_0$ as no trend, viz., $\tilde{Y}_1 = \tilde{Y}_2 = ... = \tilde{Y}_I$ and the alternative hypothesis $H_1$ as $\tilde{Y}_1 \leq \tilde{Y}_2 \leq ... \leq \tilde{Y}_I$, with at least one strict inequality. The JT statistic is based on the number $U$ of $(i < j) \in I \times I$ pairs favorable to a trend (see [7] for details). Under the hypothesis of large data set, $Z = (U - \mu_U)/\sigma_U$ is normally distributed and the single tailed JT statistic is defined as $JT = |Z|$. The JT statistic is computed from the MATLAB JTtrend.m function [3]. The significance level is set at 0.05 and the critical value is $JT_{0.05} = 1.6$.

### B. Time-domain

Fig. 3 shows a $N = 10$, $|\mathrm{OUT}\rangle = |2\rangle$, instantaneous readout (as opposed to windowed readout as in [5]) case study with $J_{45}$ uncertainty, with an error $1 - \mathcal{F}$ (as opposed to $1 - \mathcal{F}^2$ as in [5]). It confirms the anti-classical trend of concordance between error and log-sensitivity especially from controller 1 to 200. Such a trend was already observed in [5], but here it is in a context that relates better to the "tracking error," modified with a phase factor to make it relevant to quantum systems.

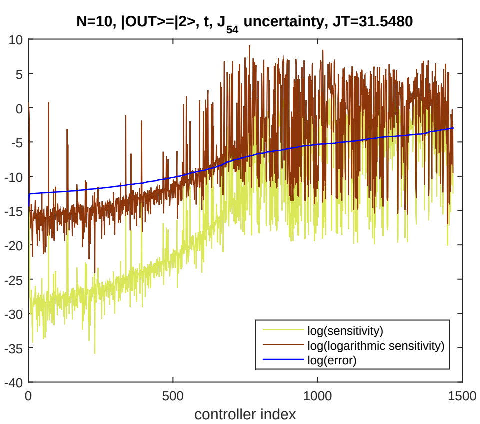


Fig. 3: Case $N = 10$, $|\mathrm{OUT}\rangle = |2\rangle$, $J_{45}$ uncertain, with $\mathcal{S}(t)$ defined by (7)

We now suppress the phase factor $e^{\jmath\phi^*}$ (remove the shaded boxes in Fig. 2) and obtain Fig. 4.

Comparing Figs. 3 and 4, it is noted that, not surprisingly, the latter error has significantly increased, because of the removal of $e^{\jmath\phi^*}$ in (6). Surprisingly, the log-sensitivity has also increased in the 1:300 range of controllers. More importantly, the latter log-sensitivity does not show an increasing trend with the error, confirmed by the JT test that accepts the $H_0$ hypothesis of no trend. "No trend" in the log-sensitivity while the error increases is rather classical.

### C. $s$-domain

The Laplace domain approach is useful to investigate asymptotic behavior, as made precise by Theorem 2. Moreover, it especially makes sense in the localization case

[3]See https://www.mathworks.com/matlabcentral/fileexchange/22159-jonckheere-terpstra-test-on-trend

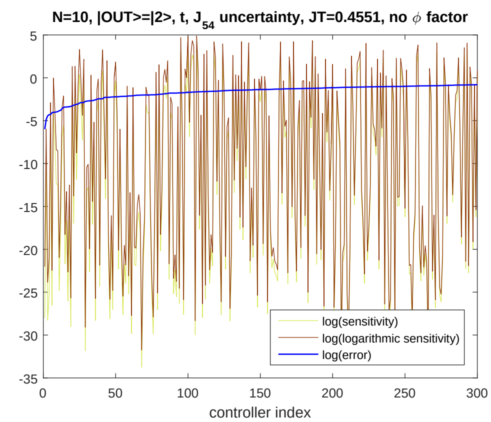


Fig. 4: Case $N = 10$, $|\mathrm{OUT}\rangle = |2\rangle$, $J_{45}$ uncertain, with $S(t)$ defined by (11)

($|\mathrm{OUT}\rangle = |\mathrm{IN}\rangle$). By symmetry, we set $|\mathrm{IN}\rangle = |1\rangle$. Numerical exploration reveals two cases:

1) The case where the spin to hold the excitation, $|\mathrm{IN}\rangle = |1\rangle$, has an uncertain coupling strength with its neighbor; by symmetry, the uncertainty is on $J_{12}$. Representatives of such case are Figs. 5-6.
2) The case where the uncertain strength $J_{mn}$ is between spins not holding the excitation; by symmetry, $m, n \neq 1$. Representatives of such case are Figs. 7-8.

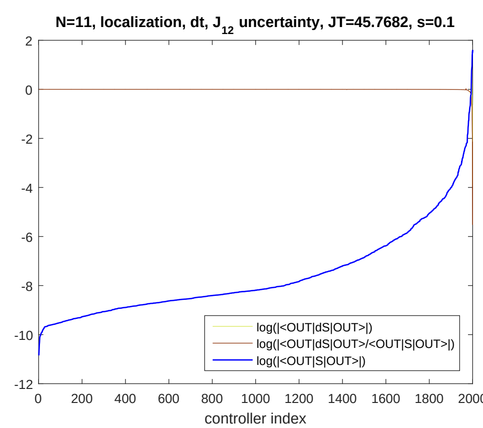


Fig. 5: Case $N = 11$, $|\mathrm{OUT}\rangle = |1\rangle$, $J_{12}$ uncertainty, $\hat{\mathcal{S}}$(s) defined as in Eq. (9), with phase factor

Common to Cases 1) and 2) is an obvious error/log-sensitivity trend reversal associated with the removal of the global phase factor. With the global phase factor of Figs. 5 and 7 one observes an anti-classical concordance between the error and the log-sensitivity. Without global phase factor, however, as shown in Figs. 6 and 8, the trend is classical–discordance between error and log-sensitivity.

Specifically in Case 1), with phase factor, the log-sensitivity is nearly "flat" at 100%, but the error is very small; without the phase factor, the trend is completely reversed; the error is "flat" and the sensitivity is significantly reduced. In Case 2), the trend reversal is the same, but not

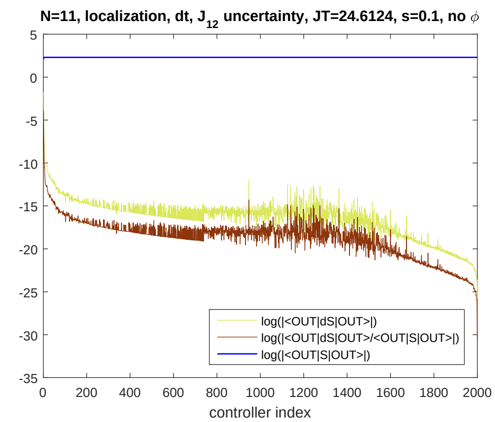


Fig. 6: Case $N = 11$, $|\mathrm{OUT}\rangle = |1\rangle$, $J_{12}$ uncertainty, $\hat{S}(s)$ defined as in Eq. (12), without phase factor

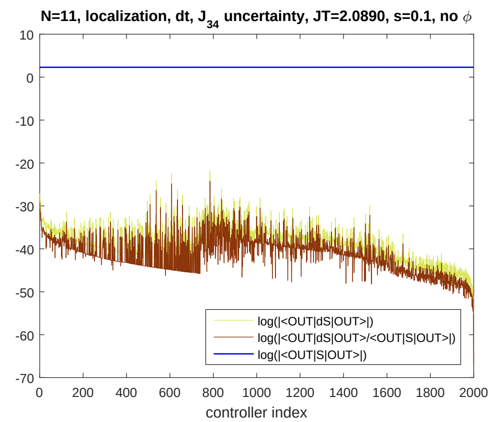


Fig. 8: Case $N = 11$, $|\mathrm{OUT}\rangle = |2\rangle$, $J_{34}$ uncertainty, $\hat{S}(s)$ defined as in Eq. (12), without phase factor

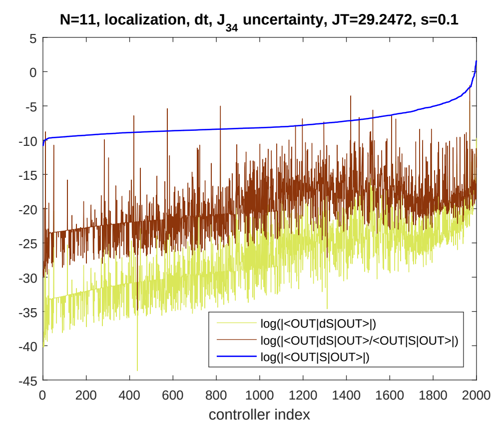


Fig. 7: Case $N = 11$, $|\mathrm{OUT}\rangle = |1\rangle$, $J_{34}$ uncertainty, $\hat{\mathcal{S}}(s)$ defined as in Eq. (9), with phase factor

as "brutal" is in Case 1). Nevertheless, with removal of the phase factor the error increases while the log-sensitivity decreases.

## VI. Conclusion

In this paper, we have studied robustness of energy landscape control for excitation transport in ring shaped quantum routers. The fundamental stumbling block in comparing classical versus quantum robustness is that energy landscape control does not fit in the paradigm of Fig. 1, which has been the basic architecture upon which classical error versus log-sensitivity limitations, e.g., $\hat{\mathtt{S}}(s) + \mathtt{T}(s) = I$, were built. While a "fictitious" loop matrix can be defined to force the architecture of Fig. 1 in landscape control, it does not factor as the cascade of a controller and an open-loop system. The closest-to-classical feedback structure to model landscape control is the one of Fig. 2, where a projective tracking error has been substituted for the classical tracking error to accommodate the quantum mechanical global phase factor shown in the shaded boxes. In this architecture, a quantum limitation $\hat{\mathcal{S}}(s) + \mathcal{T}(s) = I$ holds, but does not support the $|\mathrm{IN}\rangle$/$|\mathrm{OUT}\rangle$-selectivity of the controller. With the selectivity in place, the controller escapes the $\hat{\mathcal{S}}(s) + \mathcal{T}(s) = I$ limitation and shows some anti-classical behavior in the sense of concordance between error and log-sensitivity.

The real question that is answered here in the affirmative here is whether this anti-classical behavior is quantum mechanical. *The only way to answer such a question is to remove the quantum mechanical global phase factor from Fig. 2, which results in a complete reversal of the trends.* As our major result, this demonstrates the quantum mechanical origin of the anti-classical behavior.

Other ways to gauge the quantum effect, like those suggested in Sec. III-C, will be explored in a further paper.

## Appendix

### A. Closing the path in Eq. (10)

To compute the convolution (10), it is convenient to close the vertical integration line $(c - \jmath\infty, c + \jmath\infty)$ by a large semicircle so as to reduce the integral to residue calculations. Here we show that the path should be closed in the LHP. Indeed, remember that the proof of $\widehat{x(t)y(t)} = (\hat{X} \star \hat{Y})(s)$ proceeds from substituting $(1/(2\pi\jmath)) \int_{c-\jmath\infty}^{c+\jmath\infty} \hat{Y}(z)e^{zt}dz$ for $y(t)$ in $\widehat{x(t)y(t)} = \int_0^\infty x(t)y(t)e^{-st}dt$, where initially $(c - \jmath\infty, c + \jmath\infty)$ is in the domain of convergence of $\hat{Y}(s)$. Interchanging the order of the integrals further restricts $(c - \jmath\infty, c + \jmath\infty)$ to be in the common $z$-domain of convergence of $\hat{X}(s - z)$ and $\hat{Y}(z)$. These operations bring the factor $e^{zt}$ in the integrand; therefore, the integration of such integrand along a large semi-circle of radius $R$ in the LHP vanishes as $R \to \infty$.

### B. Laplace domain convolution: sensitivity

Many of the sensitivity formulas in the main text involve $s$-domain convolution. Here, we outline the basic computation of such convolution by residue calculations in the asymptotic case of perfect state transfer. This asymptotic study is motivated by previous analysis [7] dealing with anti-classical sensitivity properties of controllers nearly achieving the upper bound on the fidelity.

In case $|\langle \mathrm{OUT}|\Psi(t)\rangle| = 1$, the convolution $\widehat{e^{\jmath\phi^*(t)}} \star (sI + \jmath H_D)^{-1}$ reduces to $\langle \mathrm{OUT}|(sI - \jmath H_D)^{-1}|\mathrm{IN}\rangle \star (sI + \jmath H_D)^{-1}$. Set $e_i = |\mathrm{IN}\rangle$ and $e_o = |\mathrm{OUT}\rangle$ to simplify the notation; let $\{\omega_k\}_{k=1}^N$ be the (real) eigenvalues of $H_D$ with $\{\Pi_k\}_{k=1}^N$ the set of eigenprojections. With this notation, the convolution becomes

$$\begin{aligned}
&\widehat{e^{\jmath\phi^*(t)}} \star (sI + \jmath H_D)^{-1} \\
&\quad = \quad e_i^\dagger \widehat{e^{\jmath H_D t}} e_o \star (sI + \jmath H_D)^{-1} \\
&\quad = \quad e_i^\dagger (sI - \jmath H_D)^{-1} e_o \star (sI + \jmath H_D)^{-1} \\
&\quad = \quad \sum_{k\ell} \frac{e_i^\dagger \Pi_k e_o}{s - \jmath\omega_k} \star \frac{\Pi_\ell}{s + \jmath\omega_\ell} \\
&\quad = \quad \frac{1}{2\pi\jmath} \sum_{k\ell} \int_{c-\jmath\infty}^{c+\jmath\infty} \left( \frac{e_i^\dagger \Pi_k e_o}{s - z - \jmath\omega_k} \frac{\Pi_\ell}{z + \jmath\omega_\ell} \right) dz \\
&\quad = \quad \sum_{k\ell} \mathrm{Res} \left( \frac{e_i^\dagger \Pi_k e_o}{s - z - \jmath\omega_k} \frac{\Pi_\ell}{z + \jmath\omega_\ell}, z = -\jmath\omega_\ell \right) \\
&\quad = \quad \sum_{k\ell} e_i^\dagger \Pi_k e_o \frac{\Pi_\ell}{s + \jmath(\omega_\ell - \omega_k)}.
\end{aligned}$$

A few words of explanation: The third equality is just a matter of the eigendecomposition of the various operators. At the fourth equality, observe that the domain of convergence of both Laplace transforms $(sI \pm H_D)^{-1}$ is $\Re s > 0$, which implies at the 4th inequality that $\Re(s-z) > 0$ and $\Re(z) > 0$. (This secures the vanishing of both inverse Laplace transforms for $t < 0$ [12, Sec. 5.4.5].) It follows that the path of integration is a vertical line of real part $c \in (0, \Re s)$ in the common domain of definition of the two factors [2, Table 11.1]. At the fifth equality, the vertical line $(c - \jmath\infty, c + \jmath\infty)$ is closed with a large semi-circle in the left-half plane, as justified in Appendix A. The integral therefore equals the sum of the residues of $e_0^\dagger \Pi_k e_i \Pi_\ell / ((s-z) - \jmath\omega_k)(z + \jmath\omega_k))$ in $\{z : \Re z < c\}$, which contains the $z$-poles of $1/(z + \jmath\omega_k)$, but not those of $1/((s-z) - \jmath\omega_k)$.

### C. Laplace-domain convolution: differential of sensitivity

If we elect either to compute directly the sensitivity of $\mathcal{S}(s)$ relative to $H_D$ or compute it via $\mathcal{L}(s)$, we do need the sensitivity of the $s$-domain convolution $\widehat{e^{\jmath\phi^*(t)}} \star (sI + H_D)^{-1}$ relative to $H_D$, with obvious identity

$$\begin{aligned}
&d\left(\widehat{e^{\jmath\phi^*(t)}} \star (sI + \jmath H_D)^{-1}\right) \qquad (18) \\
&= \left(d\widehat{e^{\jmath\phi^*(t)}}\right) \star (sI + \jmath H_D)^{-1} + \widehat{e^{\jmath\phi^*(t)}} \star \left(d(sI + \jmath H_D)^{-1}\right).
\end{aligned}$$

First, observe that, in case of perfect state transfer,

$$d\widehat{e^{\jmath\phi^*(t)}} = \jmath \sum_{k\ell} e_i^\dagger \Pi_k dH_D \Pi_\ell e_o \frac{1}{(s - \jmath\omega_k)(s - \jmath\omega_\ell)}$$

and further

$$d(sI + \jmath H_D)^{-1} = -\jmath \sum_{k\ell} \Pi_k dH_D \Pi_\ell \frac{1}{(s + \jmath\omega_k)(s + \jmath\omega_\ell)}.$$

Therefore, for both convolutions in the sum (18), the culprit is

$$\left( \sum_{k\ell} A_{k\ell} \frac{1}{(s \mp \jmath\omega_k)(s \mp \jmath\omega_\ell)} \right) \star \left( \sum_m B_m \frac{1}{s \pm \jmath\omega_m} \right),$$

where, for the first convolution,

$$A_{k\ell} = \jmath e_i^\dagger \Pi_k dH_D \Pi_\ell e_o, \quad B_m = \Pi_m, \quad \text{upper signs}$$

and for the second convolution

$$A_{k\ell} = -\jmath \Pi_k dH_D \Pi_\ell, \quad B_m = e_i^\dagger \Pi_m e_o, \quad \text{lower signs.}$$

To compute the generic convolution, we follow the same lines as in the preceding subsection:

$$\begin{aligned}
&\sum_{k\ell} \frac{A_{k\ell}}{(s \mp \jmath\omega_k)(s \mp \jmath\omega_\ell)} \star \sum_m \frac{B_m}{s \pm \jmath\omega_m} \\
&= \frac{1}{2\pi\jmath} \sum_{k\ell m} \\
&\quad \int_{c-\jmath\infty}^{c+\jmath\infty} \frac{A_{k\ell}}{(s - z \mp \jmath\omega_k)(s - z \mp \jmath\omega_\ell)} \frac{B_m}{(z \pm \jmath\omega_m)} dz \\
&= \sum_{k\ell m} \mathrm{Res} \\
&\quad \left( \frac{A_{k\ell}}{(s - z \mp \jmath\omega_k)(s - z \mp \jmath\omega_\ell)} \frac{B_m}{z \pm \jmath\omega_m}, z = \mp\jmath\omega_m \right) \\
&= \sum_{k,\ell,m} \frac{A_{k\ell} B_m}{(s + \jmath(\mp\omega_k \pm \omega_m))(s + \jmath(\mp\omega_\ell \pm \omega_m))}.
\end{aligned}$$

### D. Explicit expressions for projective $\mathcal{S}$ and $d\mathcal{S}$

Putting together the results of Appendices B-C yields

$$\widehat{\mathcal{S}} = \left( I/s - \sum_{k\ell} (e_i^\dagger \Pi_k e_o) \frac{\Pi_\ell P}{s + \jmath(\omega_\ell - \omega_k)} \right) \qquad (19)$$

and

$$\begin{aligned}
&d\widehat{\mathcal{S}} = \\
&-\jmath \sum_{k\ell m} \frac{(e_i^\dagger \Pi_k dH_D \Pi_\ell e_o) \Pi_m P}{(s + \jmath(\omega_m - \omega_k))(s + \jmath(\omega_m - \omega_\ell))} \\
&+\jmath \sum_{k\ell m} \frac{\Pi_k dH_D \Pi_\ell P (e_i^\dagger \Pi_m e_o)}{(s + \jmath(\omega_k - \omega_m))(s + \jmath(\omega_\ell - \omega_m))}. \qquad (20)
\end{aligned}$$

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