

Effect of quantum mechanical global phase factor on error versus sensitivity limitation in quantum routing

E. Jonckheere, S. Schirmer, and F. Langbein

Abstract—In this paper, we explore the effect of the purely quantum mechanical global phase factor on the problem of controlling a ring-shaped quantum router to transfer its excitation from an initial spin to a specified target spin. “Quantum routing” on coherent spin networks is achieved by shaping the energy landscape with static bias control fields, which already results in the nonclassical feature of purely oscillatory closed-loop poles. However, more to the point, it is shown that the global phase factor requires a projective re-interpretation of the traditional tracking error where the wave function state is considered modulo its global phase factor. This results in a time-domain relaxation of the conflict between small tracking error and small sensitivity of the tracking error to structured uncertainties. While fundamentally quantum routing is achieved at a specific final time and hence calls for time-domain techniques, we also develop a projective s -domain limitation.

I. INTRODUCTION

We consider a spintronic network of N XX-coupled spins in its single excitation subspace. The latter means that one spin and one spin only is excited, “up,” while all others are “down.” In this subspace, we choose a basis such that the wave function $|\Psi\rangle = e_n$, where $\{e_n\}_{n=1}^N$ is the natural basis of \mathbb{C}^N over \mathbb{C} , denotes the quantum state where the sole excitation is on spin $\#n$. In the chosen basis, for XX-couplings, the Hamiltonian H is the adjacency matrix of the graph of the spin couplings, weighted by the coupling strengths, with zeros on the diagonal. A simple example is given by the XX-ring structure, where the Hamiltonian has tridiagonal-like structure,

$$H = \begin{pmatrix} 0 & J_{1,2} & \dots & 0 & J_{1,N} \\ J_{2,1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & J_{N-1,N} \\ J_{N,1} & 0 & \dots & J_{N,N-1} & 0 \end{pmatrix}. \quad (1)$$

In the above, $J_{m,n} = J_{n,m}$ to make the Hamiltonian Hermitian. We operate in a system of units where $\hbar = 1$ and the network has uniform couplings with strengths J_{mn} , $m \neq n$, normalized to 1.

The “open-loop” Schrödinger equation reads $|\dot{\Psi}(t)\rangle = -jH|\Psi(t)\rangle$, subject to some initial condition $|\Psi(0)\rangle = |\text{IN}\rangle$, where $|\text{IN}\rangle = e_i$ denotes the quantum state with the excitation on some “input” spin i . The control objective

is to transfer the $|\text{IN}\rangle$ state to some $|\text{OUT}\rangle = e_o$ state where the excitation is on some “output” spin o . This is to be accomplished in a short amount of time t_f and with maximum fidelity¹, $\mathcal{F}(t_f) := |\langle \text{OUT} | \Psi(t_f) \rangle|$. This is achieved by *i-o selectively* modifying the energy landscape with static bias fields $\{D_n\}_{n=1}^N$ applied to the respective spins, resulting in the Hamiltonian $H_D = H + D$, where $D = \text{diag}\{D_n\}_{n=1}^N$. The controlled Schrödinger equation becomes

$$\begin{aligned} |\dot{\Psi}(t)\rangle &= -j(H + D)|\Psi(t)\rangle, & |\Psi(0)\rangle &= |\text{IN}\rangle, \\ &= -jH|\Psi(t)\rangle + u(t), & u(t) &= -jD|\Psi(t)\rangle. \end{aligned} \quad (2)$$

It is observed that the right-hand side is split, somewhat artificially, into an open-loop term $-jH|\Psi(t)\rangle$ and a “control” term $u(t)$. Despite the *appearance* of this control as a classical measurement-mediated feedback, it *does not* need measurement of the state (and does not create back-action of the measurements); indeed, the feedback is *field-mediated* by the physical interaction between the spins and the bias fields. Nevertheless, $u(t)$ has the *mathematical* structure of a classical feedback and as such the question is whether it is subject to some of the classical error-versus-sensitivity limitations. Classically, such limitations refer to a tracking error $|\text{OUT}\rangle - |\Psi(t)\rangle$ and its sensitivity to uncertainties, but in the quantum context, the error is made smaller by considering the wave function modulo its phase factor. This paper investigates the impact of such global phase factor on the log-sensitivity of the error and points to a relaxation of the traditional conflict.

This paper follows in the footsteps of [6], where classical limitations are shown to be defeated in the time-domain, whereas here we also investigate the frequency-domain limitations. All data is from the database [10]. Proofs are omitted, but available in [7].

A. Notation

Throughout the paper, we consider three feedback configurations: the CLASSICAL configuration of Fig. 1, the QUANTUM configuration of Fig. 2 with the global phase factor shown in the shaded areas, and the semi-classical configuration of Fig. 2 but with the global phase factors removed. The relevant quantities are as follows:

- $L(s), \widehat{S}(s), T(s) := I - \widehat{S}(s)$: classical (Fig. 1) loop matrix, sensitivity and complementary sensitivity matrices, resp.

¹Sometimes the fidelity is defined as $|\langle \text{OUT} | \Psi(t_f) \rangle|^2$.

E. Jonckheere is with the Dept. of Elec. and Comp. Eng., Univ. of Southern California, jonckhee@usc.edu

S. Schirmer is with the Physics Dept., Singleton Campus, Swansea University, UK, lw1660@gmail.com

F. Langbein is with the School of Computer Science and Informatics, Cardiff University, UK, langbeinf@cardiff.ac.uk

- $\mathcal{L}(s)$, $\hat{\mathcal{S}}(s)$, $\mathcal{T}(s) = I - \mathcal{S}(s)$: projective loop matrix, sensitivity and complementary sensitivity matrices, resp., with global phase factor (shaded boxes in Fig 2). $\hat{\mathcal{S}}(s)$ is defined analytic in $\Re s > 0$.
- $\mathcal{S}(t)$: inverse Laplace transform of $\hat{\mathcal{S}}(s)$, vanishing for $t < 0$.
- $L(s)$, $\hat{S}(s)$, $T(s) = I - \hat{S}(s)$: loop matrix, sensitivity and complementary sensitivity matrices, resp., without global phase factor (after removal of shaded boxes in Fig 2). $\hat{S}(s)$ is analytic in $\Re s > 0$.
- $S(t)$: inverse transform of $\hat{S}(s)$ with $S(t < 0) = 0$.

II. CLASSICAL VERSUS QUANTUM ARCHITECTURE

Laplace domain techniques are of limited use in quantum control as most of the fidelity specification are rather in the time domain. Nevertheless, as shown in [5], Laplace techniques are still useful to study steady-state ($s \approx 0$) behavior. Besides, a quick review of the Laplace domain limitations are necessary to explore the classical-quantum discrepancies.

A. Classical

The fundamental limitation looked at in the present paper is the quantum mechanical equivalent, if any, of $\hat{\mathcal{S}}(s) + \mathcal{T}(s) = I$, where $\hat{\mathcal{S}}(s) = (I + \mathcal{L}(s))^{-1}$ is the sensitivity matrix, $\mathcal{L}(s)$ is the loop matrix, and $\mathcal{T}(s)$ is the complementary sensitivity $\mathcal{L}(s)(I + \mathcal{L}(s))^{-1}$ of the classical loop shown in Fig. 1.

Note that the disturbance $\hat{r}(s)$ could be anything and does not support the notion of *selectivity*, that is, when $\hat{r}(s)$ is restricted to be a terminal target as shown in Fig. 2, nor does Fig. 1 support the initial condition $|\text{IN}\rangle$ of Fig. 2.

Given the classical tracking error $e(t) = r(t) - y(t)$, we have $\hat{e}(s) = \hat{\mathcal{S}}(s)\hat{r}(s)$ indicating that $\hat{\mathcal{S}}(s)$ is the transmission from the disturbance $\hat{r}(s)$ to the error $\hat{e}(s)$. $\mathcal{T}(s)$ on the other hand is related to the log-sensitivity of $\hat{\mathcal{S}}(s)$ to errors $d\mathcal{L}(s)$ in the loop matrix. Precisely,

$$\hat{\mathcal{S}}^{-1}(s)d\hat{\mathcal{S}}(s) = -((d\mathcal{L}(s))\mathcal{L}^{-1}(s))\mathcal{T}(s).$$

$\hat{\mathcal{S}}(s) + \mathcal{T}(s) = I$ hence quantifies the well known conflict between achieving simultaneously small tracking error and small log-sensitivity of tracking error to uncertainties, *disregarding selectivity*. If $d\mathcal{L}$ is structured to represent an uncertainty on a parameter, say J , then the above is rewritten as

$$\frac{d \ln \hat{\mathcal{S}}(s)}{d \ln J} = -\frac{d \ln \mathcal{L}(s)}{d \ln J} \mathcal{T}(s), \quad (3)$$

where $d \ln \hat{\mathcal{S}} = \hat{\mathcal{S}}^{-1}(d\hat{\mathcal{S}})$ and $d \ln \mathcal{L} = (d\mathcal{L})\mathcal{L}^{-1}$. In either case, it is observed that $\mathcal{T}(s)$ is related to the log-sensitivity of $\hat{\mathcal{S}}(s)$.

B. Phase factor and complex projective space $\mathbb{C}\mathbb{P}^{N-1}$

In the quantum control problem of moving the system from one quantum state to another one, there is no tracking error to be minimized, but a fidelity $\mathcal{F}(t_f) = |\langle \text{OUT} | \Psi(t_f) \rangle|$ to be maximized relative to D . However, the maximization

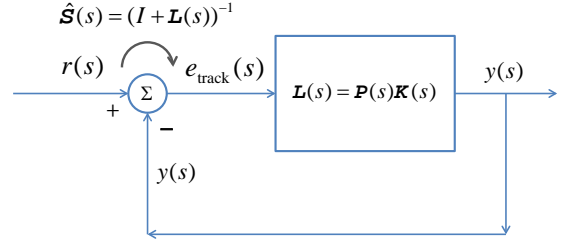


Fig. 1: Classical single-degree-of-freedom loop

of the fidelity can be related to minimization of a tracking error understood in some projective sense.

Theorem 1: The optimal controller achieving the maximum fidelity

$$\max_D |\langle \text{OUT} | \Psi(t_f) \rangle|$$

is the same as the controller achieving

$$\min_D \left(\min_{\phi} \| |\text{OUT}\rangle - e^{j\phi} |\Psi(t_f)\rangle \| \right), \quad (4)$$

where ϕ is a global phase factor with the minimum achieved for

$$\phi^*(t_f) = -\angle \langle \text{OUT} | \Psi(t_f) \rangle, \quad e^{j\phi^*(t_f)} = \frac{\langle \text{OUT} | \Psi(t_f) \rangle^\dagger}{|\langle \text{OUT} | \Psi(t_f) \rangle|}. \quad (5)$$

The preceding theorem states that controllers can as well be optimized (although in a somewhat computationally clumsy way) on the basis of the *projective tracking error*

$$e_{\text{proj}}(t) = |\text{OUT}\rangle - e^{j\phi^*(t)} |\Psi(t)\rangle \quad (6)$$

with the already perceived reward that the above connects with classical concepts.

More formally speaking, since $\| |\Psi\rangle \|_{\mathbb{C}^N} = 1$ and since a phase factor $\exp(-j\phi)$ does not fundamentally change the quantum state, $|\Psi\rangle$ lives in $\mathbb{S}^{2N-1}/\mathbb{S}^1 = \mathbb{C}\mathbb{P}^{N-1}$, the complex projective space. Observe that the fidelity $|\langle \text{OUT} | \Psi(t_f) \rangle|$ is the cosine of the Fubini-Study metric on $\mathbb{C}\mathbb{P}^{N-1}$. More closely related to (6), observe the following:

Corollary 1: $\delta(|\text{OUT}\rangle, |\Psi\rangle) := \min_{\phi} \| |\text{OUT}\rangle - e^{-j\phi} |\Psi\rangle \|_{\mathbb{C}^N}$ is a metric on $\mathbb{C}\mathbb{P}^{N-1}$.

Remark 1: The global phase $\phi^*(t)$ could be viewed as an ad hoc trick to think maximum fidelity in terms of δ -minimum tracking error. However, for it to have its classical quantum mechanical interpretation, it needs to be constant, which could be accomplished by limiting it to $\phi^*(t_f)$. However, a time-varying global phase $\phi^*(t)$ could have the quantum mechanical interpretation of change of the zero energy level. Indeed, a shift of energy level $H_D \rightarrow H_D + cI$ yields a phase factor $\exp(-jct)$. From (5), under near perfect state transfer, it follows that this specific global phase factor could be associated with a shift $c = \langle \text{OUT} | H_D | \text{IN} \rangle$.

C. Projective sensitivity

Observing from (2) that $|\Psi(t)\rangle = e^{-jH_D t} |\text{IN}\rangle$ and defining the output-input swapping operator

$$P = |\text{IN}\rangle \langle \text{OUT}|$$

the projective tracking error leads to the concept of *projective sensitivity function* $\mathcal{S}(t)$,

$$e_{\text{proj}}(t) = \underbrace{\left(I - e^{j\phi^*(t)} e^{-jH_D t} P \right)}_{\mathcal{S}(t)} |\text{OUT}\rangle. \quad (7)$$

The connection with the classical relationship $\hat{e}(s) = \hat{S}(s)\hat{r}(s)$ is obvious, but note the selectivity feature of the above that the disturbance $|\text{OUT}\rangle$ is selectively restricted to be a natural basis of \mathbb{C}^N . In fact, the selectivity is 2-fold, as contrary to a classical controller, D is not universal, as it is selectively optimized for $|\text{OUT}\rangle$. To connect the above with the fidelity, observe that

$$\begin{aligned} \langle \text{OUT} | \mathcal{S}(t) | \text{OUT} \rangle &= \langle \text{OUT} | e_{\text{proj}}(t) \rangle \\ &= 1 - (\langle \text{OUT} | e^{-jH_D t} | \text{IN} \rangle) e^{j\phi^*(t)} \\ &= 1 - \mathcal{F}(t), \end{aligned} \quad (8)$$

where the third equality is seen by remembering that $\phi^*(t)$ is chosen so as to make $(\langle \text{OUT} | e^{-jH_D t} | \text{IN} \rangle) e^{j\phi^*(t)}$ real and positive.

Here we are at the crucial point. Even though quantum transport is usually formulated in terms of fidelity, Eq. (8) reveals that we could equally argue in terms of the *projective time-sensitivity function* $\mathcal{S}(t)$.

Fidelity is usually formulated as above in the time-domain; however, Laplace domain techniques have also been used [5] but in the very specific context of steady-state behavior ($s \approx 0$). Nevertheless, to better connect with the classical concepts, usually formulated in the Laplace domain, we define $\hat{\mathcal{S}}(s)$ via

$$\hat{e}_{\text{proj}}(s) = \underbrace{\left(I/s - \widehat{e^{j\phi^*(t)}} \star (sI + jH_D)^{-1} P \right)}_{\hat{\mathcal{S}}(s)} |\text{OUT}\rangle, \quad (9)$$

where the widehat notation denotes the unilateral Laplace transform and \star denotes the Laplace domain convolution

$$(\hat{X} \star \hat{Y})(s) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \hat{X}(s-z)\hat{Y}(z)dz, \quad (10)$$

where the integration path is a vertical line in the common z -domain of convergence of $\hat{X}(s-z)$ and $\hat{Y}(z)$, assuming such a nonempty intersection exists. Relevant results are summarized in Appendix A.

The problem is that $\hat{\mathcal{S}}(s)$ does not naturally lend itself to a representation of the form $(I + \mathcal{L}(s))^{-1}$ with the idea that $\mathcal{L}(s)$ factors as $\mathcal{P}(s)\mathcal{K}(s)$, where $\mathcal{P}(s)$ is some plant and $\mathcal{K}(s)$ some controller. At best, $\hat{\mathcal{S}}(s)$ can be related to the architecture shown in Fig. 2, which is certainly not of the single degree of freedom configuration, but could be interpreted as a 3-degree of freedom one, notwithstanding the feedbacks involved in the phase function. Following the

classical path of ideas, we define a fictitious loop matrix \mathcal{L} to reproduce the classical relation $\hat{\mathcal{S}}(s) = (I + \mathcal{L})^{-1}(s)$, that is, $\mathcal{L} = \hat{\mathcal{S}}^{-1} - I$; explicitly,

$$\mathcal{L}(s) = \left(I/s - \widehat{e^{j\phi^*(t)}} \star (sI + jH_D)^{-1} P \right)^{-1} - I.$$

D. Classical oscillatory systems

Schrödinger's equation (2) is, after all, an Ordinary Differential Equation (ODE) over \mathbb{C}^n and should the eigenvalues of jH_D come in complex conjugate pairs, it could be interpreted as a lossless spring mechanical system or a LC oscillatory circuit. Moreover, “energy landscape” techniques have been popular in robotics and electromechanical systems [8], [11], where the energy is shaped so as to put its minimum at the target by *local* feedbacks bearing similarity with $u_k = -jD_k\Psi_k$. Such classical systems follow the architecture of Fig. 2—without the global phase factors in the shaded areas—with relevant tracking error defined as, reverting to classical notation,

$$e(t) = \underbrace{\left(I - e^{-jH_D t} P \right)}_{\mathcal{S}(t)} |\text{OUT}\rangle, \quad (11)$$

or taking the unilateral Laplace transform

$$\hat{e}(s) = \underbrace{\left(I/s - (sI + jH_D)^{-1} P \right)}_{\hat{\mathcal{S}}(s)} |\text{OUT}\rangle. \quad (12)$$

The importance of this case-study is that comparison between the two sensitivity matrices $\hat{\mathcal{S}}(s)$ and $\hat{S}(s)$ would narrow down quantum enhancement, if any, regarding circumventing the classical limitations. This is essentially what is addressed in Sec. V-B.

III. SENSITIVITY—LAPLACE DOMAIN

A. Selective sensitivity

Taking the Laplace transform of (8) and using (9) yields

$$\langle \text{OUT} | \hat{\mathcal{S}}(s) | \text{OUT} \rangle = 1/s - \hat{\mathcal{F}}(s).$$

Taking the log-differential, while remembering that nominally $J_{mn} = 1$, yields

$$\frac{d\langle \text{OUT} | \hat{\mathcal{S}}(s) | \text{OUT} \rangle}{dJ_{mn}} \frac{1}{\langle \text{OUT} | \hat{\mathcal{S}}(s) | \text{OUT} \rangle} \quad (13)$$

$$= - \left\langle \text{OUT} \left| \frac{d\hat{\mathcal{F}}(s)}{dJ_{mn}} \right| \text{OUT} \right\rangle \frac{1}{1/s - \hat{\mathcal{F}}(s)}, \quad (14)$$

where the expressions for $\hat{\mathcal{S}}(s)$ and $d\hat{\mathcal{S}}(s)$ are summarized in Appendix A. Such quantities are numerically explored in Sec. V-B.

B. Motivation for Laplace techniques: Asymptotic results

Here we provide motivation for the sensitivity analysis of $\hat{\mathcal{S}}(s)$. We proceed from the explicit expressions for $\hat{\mathcal{S}}(s)$ and $d\hat{\mathcal{S}}(s)$ of Appendix A and use a generalized Laplace final value theorem to derive some asymptotic behavior of $\mathcal{S}(t)$, $d\mathcal{S}(t)$ as $t \rightarrow \infty$. Moreover, in the quest for a quantum enhancement, we contrast those results with the limiting

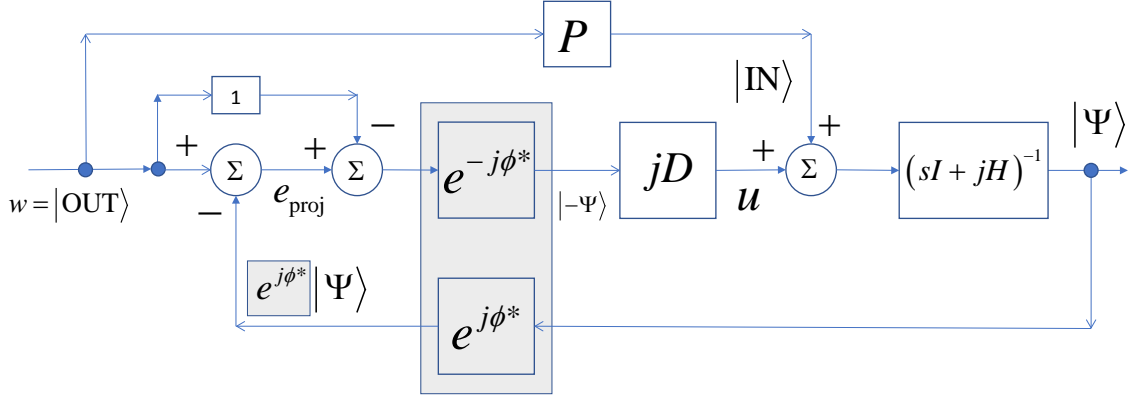


Fig. 2: The projective error e_{proj} embedded in a semi-classical 3-degree-of-freedom loop. The top paths (P and 1) are indeed two additional degrees of freedom relative to the single degree of freedom configuration. The shaded areas refer to the “global phase factor.” Note that the $e^{\pm j\phi^*}$ operations have to be interpreted in the time-domain.

behavior of $S(t)$, $dS(t)$ when they do not include the global phase factor (no shaded boxes in Fig. 2).

Since our systems are not closed-loop stable in the classical sense, we need a generalized Laplace final value theorem ²:

Theorem 2: Nonclassical Laplace final value theorem [3, Th. 2]. Let $\hat{f}(s)$ be the Laplace transform of $f(t)$. If $\lim_{s \rightarrow 0} \int_s^\infty (\hat{f}(\xi)/\xi) d\xi = \infty$, then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\tau) d\tau = \lim_{s \rightarrow 0} s \hat{f}(s). \quad \blacksquare$$

In the following, we highlight the difference between the two cases: with and without global phase factor (with and without shaded boxes in Fig. 2) as a way to gauge quantum effects.

Theorem 3: Regarding the average steady-state error in the sense of Th. 3, we have

1) *With global phase factor:*

$$\lim_{s \downarrow 0} \langle \text{OUT} | s \hat{S}(s) | \text{OUT} \rangle = 1 - \sum_k |\langle \text{OUT} | \Pi_k | \text{IN} \rangle|^2.$$

2) *Without global phase factor:*

$$\begin{aligned} \lim_{s \downarrow 0} \langle \text{OUT} | s \hat{S}(s) | \text{OUT} \rangle &= 1 - \langle \text{OUT} | \text{IN} \rangle, \\ &= \begin{cases} 1 & \text{for transfer,} \\ 0 & \text{for localization.} \end{cases} \end{aligned}$$

Theorem 4: For the differential $\langle \text{OUT} | s d\hat{S}(s) | \text{OUT} \rangle$, we have

1) *With global phase factor:*

$$\lim_{s \downarrow 0} \langle \text{OUT} | s d\hat{S}(s) | \text{OUT} \rangle = \begin{cases} \infty & \text{for transfer,} \\ 0 & \text{for localization.} \end{cases}$$

²Ph. Anderson in his famous localization paper [1] was aware of and utilized this result, but not with the level of rigor as in [3].

2) *Without global phase factor and with $\lambda_k(H_D) \neq 0$:*

$$\lim_{s \downarrow 0} \langle \text{OUT} | s d\hat{S}(s) | \text{OUT} \rangle = 0.$$

C. Comparison with classical, nonselective sensitivity

Because the relationship between \mathcal{L} and \hat{S} is the same as that between $\hat{L}(s)$ and $\mathcal{S}(s)$, the log-sensitivity of $\hat{S}(s)$ with respect to coupling parameters in \mathcal{L} is structurally the same as (3),

$$\frac{d \ln \hat{S}(s)}{d \ln J_{mn}} = - \frac{d \ln \mathcal{L}(s)}{d \ln J_{mn}} \mathcal{T}(s), \quad (15)$$

where $\mathcal{T}(s) = \mathcal{L}(s)(I + \mathcal{L}(s))^{-1}$. The above comes together with the obvious relationship

$$\hat{S}(s) + \mathcal{T}(s) = I. \quad (16)$$

The above might be called the quantum mechanical error versus sensitivity limitation, with the caveat that it does not support the selectivity of the quantum transport.

IV. SENSITIVITY—TIME DOMAIN

The starting point of the time-domain analysis is the sensitivity of the matrix exponential to variation in the matrix exponent, as given by the Zassenhaus formula [2]:

$$\begin{aligned} \exp(-j(H_D + dH_D)t) &= \\ &= \exp(-jH_D t) \exp(-jdH_D t) \\ &\quad \times \prod_{p=2}^{\infty} \exp(Z_p(H_D, dH_D)(-jt)^p), \end{aligned}$$

where $Z_p(\cdot, \cdot)$ is a homogeneous Lie polynomial of degree p , and the decomposition is unique. Note that $Z_p(H_D, dH_D)$ contains a linear term in dH_D , which should be taken into consideration when computing sensitivity. Explicitly,

$$\begin{aligned} e^{-j(H_D + dH_D)t} &= e^{-jH_D t} e^{-jdH_D t} e^{\frac{1}{2}[H_D, dH_D]t^2} \\ &\quad \times e^{\frac{t^3}{6}[[H_D, dH_D], H_D]} \\ &\quad \times e^{-\frac{t^4}{24}[[[H_D, dH_D], H_D], H_D]} \dots \end{aligned}$$

Setting $dH_D = dJ_{mn}S_{mn}$, where dJ_{mn} is the variation of the parameter J_{mn} and S_{mn} the associated structure and utilizing the above formula with its expansion restricted to include polynomials up to Z_2 yields

$$\frac{de^{-jH_D t}}{dJ_{mn}} \approx e^{-jH_D t} \left(-jS_{mn}t + \frac{1}{2}[H_D, S_{mn}]t^2 \right). \quad (17)$$

While approximate, this formula has the merit that it reveals the role of the commutator $[H_D, S_{mn}]$.

From (8), the time-domain log-sensitivity is set up as

$$\frac{d(1 - \mathcal{F})}{dJ_{mn}} \frac{1}{1 - \mathcal{F}} = - \frac{\langle \text{OUT} | \frac{dS(t)}{dJ_{mn}} | \text{OUT} \rangle}{\langle \text{OUT} | S(t) | \text{OUT} \rangle}, \quad (18)$$

where $dS(t)/dJ_{mn}$ is computed from Eqs. (7), (17), and $e^{j\phi^*(t)}$ is evaluated as given by Th. 1. Note that for numerical computations, Eq. (17) might not be accurate enough, in which case we have to revert to [12, Eq. 32]. The details are left out.

V. NUMERICAL RESULTS

A. Time-domain

Fig. 3 shows a $N = 10$, $|\text{OUT}\rangle = |2\rangle$, instantaneous readout (as opposed to windowed readout as in [4]) case study with J_{45} uncertainty, with an error $1 - \mathcal{F}$ (as opposed to $1 - \mathcal{F}^2$ as in [4]). It confirms the anti-classical trend of concordance between error and log-sensitivity especially from controller 1 to 200. Such a trend was already observed in [4], but here it is in a context that relates better to the “tracking error,” modified with a phase factor to make it relevant to quantum systems.

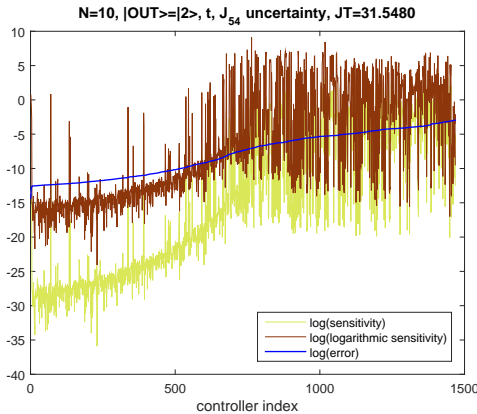


Fig. 3: Case $N = 10$, $|\text{OUT}\rangle = |2\rangle$, J_{45} uncertain, with $S(t)$ defined by (7)

We now suppress the phase factor $e^{j\phi^*}$ (remove the shaded boxes in Fig. 2) and obtain Fig. 4.

Comparing Figs. 3 and 4, it is noted that, not surprisingly, the latter error has significantly increased, because of the removal of $e^{j\phi^*}$ in (6). Surprisingly, the log-sensitivity has also increased in the 1:300 range of controllers. More importantly, the latter log-sensitivity does not show an increasing trend with the error, confirmed by the Jonckheere-Terpstra

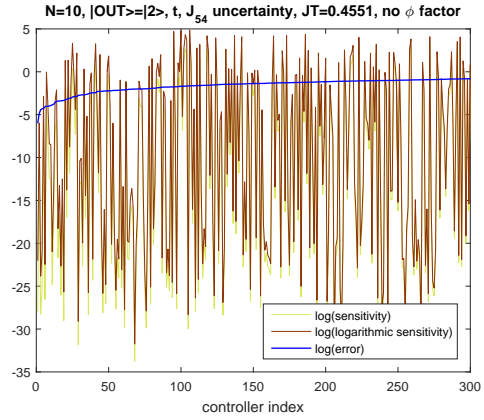


Fig. 4: Case $N = 10$, $|\text{OUT}\rangle = |2\rangle$, J_{45} uncertain, with $S(t)$ defined by (11)

(JT) test [6] that accepts the H_0 hypothesis of no trend. “No trend” in the log-sensitivity while the error increases is rather classical.

B. s-domain

The Laplace domain approach is useful to investigate asymptotic behavior, as made precise by Theorem 2. Moreover, it especially makes sense in the localization case ($|\text{OUT}\rangle = |\text{IN}\rangle$). By symmetry, we set $|\text{IN}\rangle = |1\rangle$. Numerical exploration reveals two cases:

- 1) The case where the spin to hold the excitation, $|\text{IN}\rangle = |1\rangle$, has an uncertain coupling strength with its neighbor; by symmetry, the uncertainty is on J_{12} . Representatives of such case are Figs. 5-6.
- 2) The case where the uncertain strength J_{mn} is between spins not holding the excitation; by symmetry, $m, n \neq 1$.

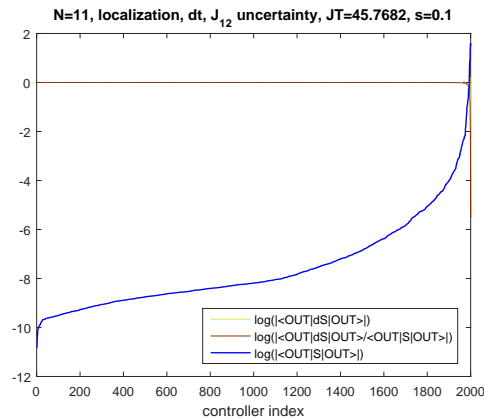


Fig. 5: Case $N = 11$, $|\text{OUT}\rangle = |1\rangle$, J_{12} uncertainty, $\hat{S}(s)$ defined as in Eq. (9), with phase factor

Common to Cases 1) and 2) is an error/log-sensitivity trend reversal associated with the removal of the global phase factor, obvious from comparing Figs. 5 and 6.

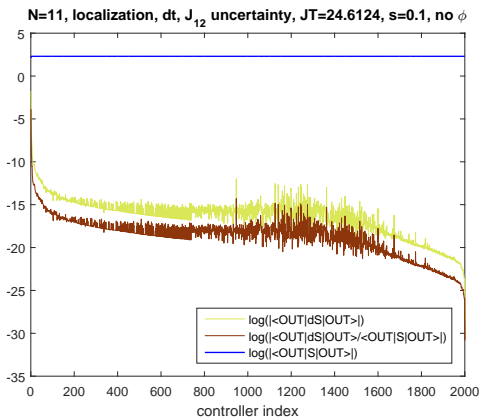


Fig. 6: Case $N = 11$, $|\text{OUT}\rangle = |1\rangle$, J_{12} uncertainty, $\hat{S}(s)$ defined as in Eq. (12), without phase factor

Specifically in Case 1), with phase factor, the log-sensitivity is nearly “flat” at 100%, but the error is very small; without the phase factor, the trend is completely reversed; the error is “flat” and the sensitivity is significantly reduced. However, a close inspection of Fig. 5 around controllers #1900-2000 reveals an abrupt increase of the sensitivity together with a decrease of the log-sensitivity, a manifestation of the projective limitation (16). In Case 2), the trend reversal is the same, but not as “brutal” is in Case 1). Nevertheless, with removal of the phase factor the error increases while the log-sensitivity decreases.

VI. CONCLUSION

In this paper, we have studied robustness of energy landscape control for excitation transport in ring shaped quantum routers. The fundamental stumbling block in comparing classical versus quantum robustness is that energy landscape control does not fit in the paradigm of Fig. 1, which has been the basic architecture upon which classical error versus log-sensitivity limitations, e.g., $\hat{S}(s) + \mathcal{T}(s) = I$, were built. The closest-to-classical feedback structure to model landscape control is the one of Fig. 2, where a projective tracking error has been substituted for the classical tracking error to accommodate the quantum mechanical global phase factor shown in the shaded boxes. In this architecture, a quantum limitation $\hat{S}(s) + \mathcal{T}(s) = I$ holds, and manifested itself at the extreme right of Fig. 5.

The real question that is answered here in the affirmative here is whether the anti-classical behavior observed in the time-domain is quantum mechanical. *The only way to answer such a question is to remove the quantum mechanical global phase factor from Fig. 2, which results in a complete reversal of the trends.* As our major result, this demonstrates the quantum mechanical origin of the anti-classical behavior.

Other ways to gauge the quantum effect, like those suggested in Sec. III-C, will be explored in a further paper.

APPENDIX

A. Explicit expressions for projective \hat{S} and $d\hat{S}$

The details of the convolutions (9)-(10) are left out, but available in [7]. The relevant results are the following:

$$\hat{S} = \left(I/s - \sum_{k\ell} (e_i^\dagger \Pi_k e_o) \frac{\Pi_\ell P}{s + j(\omega_\ell - \omega_k)} \right) \quad (19)$$

and upon differentiating \hat{S} as defined by (9),

$$d\hat{S} = -j \sum_{k\ell m} \frac{(e_i^\dagger \Pi_k dH_D \Pi_\ell e_o) \Pi_m P}{(s + j(\omega_m - \omega_k))(s + j(\omega_m - \omega_\ell))} + j \sum_{k\ell m} \frac{\Pi_k dH_D \Pi_\ell P (e_i^\dagger \Pi_m e_o)}{(s + j(\omega_k - \omega_m))(s + j(\omega_\ell - \omega_m))}. \quad (20)$$

REFERENCES

- [1] P. W. Anderson. Absence of diffusion in certain random lattices. *Physical Review*, 109(5):1492–1505, March 1958.
- [2] F. Casas, A. Murua, and Mladen Nadinic. Efficient computation of the Zassenhaus formula. *Computer Physics Communications*, 183(11):2386–2391, November 2012. Available at arXiv:1204.0389v2 [math-ph] 15 Jun 2012.
- [3] E. Gluskin and S. Miller. On the recovery of the time average of continuous and discrete time functions from their Laplace and z-transform. *International Journal of Circuit Theory and Applications*, 41(9):988–997, 2013. doi: 10.1002/cta.1877, arXiv:1109.3356v4 [math-ph].
- [4] A. Jonckheere. A distribution-free k -sample test against ordered alternatives. *Biometrika*, 41:133145, 1954. doi:10.2307/233301.
- [5] E. Jonckheere, S. Schirmer, and F. Langbein. Structured singular value analysis for spintronics network information transfer control. *IEEE Transactions on Automatic Control*, 62(12):6568–6574, December 2017. Available at arXiv:1706.03247v1 [quant-ph] 10 Jun 2017.
- [6] E. Jonckheere, S. Schirmer, and F. Langbein. Jonckheere-Terpstra test for nonclassical error versus log-sensitivity relationship of quantum spin network controllers. *International Journal of Robust and Nonlinear Control*, 28(6):2383–2403, April 2018. Available at arXiv:1612.02784 [math.OC].
- [7] E. Jonckheere, S. Schirmer, and F. Langbein. Effect of quantum mechanical global phase factor on error versus sensitivity limitation in quantum routing. Full paper, available at <http://ee.usc.edu/jonckhee>, 2019.
- [8] E. A. Jonckheere. Lagrangian theory of large scale systems. In *European Conference on Circuit Theory and Design*, pages 626–629, The Hague, the Netherlands, August 25-28 1981. invited paper.
- [9] M. G. Kendall. A new measure of rank correlation. *Biometrika*, 30(1-2):81–93, 1938.
- [10] Frank C. Langbein, Sophie G. Schirmer, and Edmond Jonckheere. Static bias controllers for XX spin-1/2 rings. Data set, figshare, DOI:10.6084/m9.figshare.3485240.v1, July 3 2016.
- [11] R. Ortega, A. Loria, Per J. Nicklasson, and H. Sira-Ramirez. *Passivity-based control of Euler-Lagrange systems: Mechanical, electrical and electromechanical applications*. Springer, London, Berlin, Heidelberg, ..., 1998.
- [12] S. Schirmer, E. Jonckheere, and F. Langbein. Design of feedback control laws for spintronics networks. *IEEE Transactions on Automatic Control*, 63(8):2523–2536, August 2018. Available at arXiv:1607.05294.
- [13] T. J. Terpstra. The asymptotic normality and consistency of Kendall’s test against trend, when ties are present in one ranking. *Indagationes Mathematicae*, 14:327333, 1952.
- [14] J. von Neumann. Distribution of the ratio of the mean square successive difference to the variance. *Ann. Math. Statist.*, 12(4):367–395, 1941.
- [15] J. von Neumann, R. H. Kent, H. R. Bellinson, and B. I. Hart. The mean square successive difference. *The Annals of Mathematical Statistics*, 12:153–162, 1941.